Towards tensor products

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Let $G_n = GL_n$, defined over a number field k. One expects a functorial lift of automorphic representations from $H = G_n \times G_m$ to $G = G_{nm}$ satisfying the relation

$$L(s, \pi_n \boxtimes \pi_m, \mathrm{std}) \stackrel{?}{=} L(s, \pi_n \times \pi_m, \otimes) = L(s, \pi_n \times \pi_m)$$

where on the left is the standard automorphic L-function on G, and on the right is the Rankin-Selberg L-function on H. By the Local Langlands correspondence for GL_n one knows that $\pi_n \boxtimes \pi_m$ can be defined as an *admissible* representation, and it remains to show that it is moreover *automorphic*. It is well known that by work of Ramakrishnan and Kim-Shahidi, the lift holds for $n = m = 2$ and $n = 2, m = 3$ respectively, using the converse theorem.

In this writeup we investigate this lift using the trace formula. The work of Ngô on the fundamental lemma of Langlands and Shelstad interprets the geometric side of the trace formula by a certain Hitchin fibration in positive characteristic. This is an attempt to develop a trace identity in this spirit.

1 Introduction

1.1 Conjugacy

We say γ_1 and γ_2 in $G_n(k)$ are stably conjugate if they are conjugate over $G_n(\overline{k})$.

Lemma 1.1.1. *Stable conjugacy and ordinary conjugacy are equivalent in G_n, also $G_n \times G_m$.*

This a well-known property of G_n; one should be able to prove this using triviality of $H^1(k, G_n)$.

The tensor product for matrices can be described explicitly as the Kronecker product. Given any two matrices A, B, define

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

One checks that the eigenvalues of $A \otimes B$ are the products of the eigenvalues of A and B.

Lemma 1.1.2. *If $\gamma_1 \otimes \gamma_2$ is a regular element in G then γ_1, γ_2 are regular in G_n and G_m respectively.*

This is clear when $\gamma_1 \otimes \gamma_2$ is regular semisimple. I haven't checked this for regular unipotent.

Consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$. Fix a Cartan subalgebra $\mathfrak{t}$ of diagonal matrices and Weyl group $W = S_n$. By Chevalley, the adjoint action of G on $\mathfrak{g}$ (i.e., conjugation) induces an isomorphism

$$k[\mathfrak{g}]^G \xrightarrow{\sim} k[\mathfrak{t}]^W$$

Let $\mathfrak{c} := \mathrm{Spec}(k[\mathfrak{t}]^W)$ be the affine space of degree n monic polynomials, isomorphic to $\mathbf{A}^n$ by

$$p_A := X^n - a_1 X^{n-1} + \cdots + (-1)^n a_n \mapsto (a_1, \ldots, a_n).$$

The *Chevalley characteristic* $\chi : \mathfrak{g} \to \mathfrak{c}$ sends A to its characteristic polynomial $\det(XI - A)$, or equivalently,

$$\chi : A \mapsto a = (\mathrm{tr}(A), \mathrm{tr}(\wedge^2 A), \ldots, \mathrm{tr}(\wedge^n A)).$$

The *Kostant section* ϵ of the map χ constructs a matrix with the given characteristic polynomial. The centralizer of a companion matrix can be identified with the centralizer of X in $GL(R)$, and one checks that

$$J_a \simeq R^\times \subset GL(R) \text{ and } I_\gamma \simeq R \subset \mathfrak{gl}(R).$$

More generally, if two regular elements in $\mathfrak{g}$ have the same characteristic polynomial a, then their centralizers are canonically isomorphic. Also, for every a there is a regular centralizer J_a isomorphic to I_γ for any γ such that $\chi(\gamma) = a$.

Obtain a *correspondence* of conjugacy classes as follows: compose the Chevalley characteristic with the tensor product (using the resultant),

$$\otimes : (a, b) \to a \otimes b$$

followed by the Kostant section ϵ, thus

$$\mathfrak{g}_1 \times \mathfrak{g}_2 \xrightarrow{\chi} \mathfrak{t}_1/W \times \mathfrak{t}_2/W \xrightarrow{\otimes} \mathfrak{t}_1/W \otimes \mathfrak{t}_2/W \subset \mathfrak{t}/W \xrightarrow{\epsilon} \mathfrak{g}$$

and call this correspondence $(\gamma_1, \gamma_2) \leftrightarrow \gamma_{12}$. Note that this is on the level of conjugacy classes.

Remark 1.1.3. Let $n = m = 2$, and denote the coefficients of the characteristic polynomials p_A and p_B by (a_1, a_2) and (b_1, b_2). The coefficients of $p_{A \otimes B}$ are $(c_1, c_2, c_3, c_4) = (a_1 b_1, a_1^2 b_2 + b_1^2 a_2 - 2a_2 b_2, b_1 b_2 a_1 a_2, a_2^2 b_2^2)$.

In general every c_i can be expressed in terms of elementary symmetric functions of a_j and b_k for all n, m using the resultant of the two polynomials. Namely, one has

$$p_{A \otimes B}(t) = \mathrm{res}(p_A(x), p_B(t/x) x^m)$$

where the resultant taken with with respect to the indeterminate x. Alternatively, the c_i can also sometimes be described using the theory of λ-rings.

1.2 Orbital integrals

Let k be a local field. Define the local orbital integral of a smooth compactly supported f on $G(k)$

$$\mathbf{O}_\gamma(f) = \int_{I_\gamma(k)\backslash G(k)} f(g^{-1}\gamma g)dg$$

and its weighted version

$$J_M^G(\gamma, f) = |D^G(\gamma)|^{\frac{1}{2}} \int_{I_\gamma(k)\backslash G(k)} f(g^{-1}\gamma g) v_M^G(g)dg.$$

where $D^G(\gamma)$ is the Weyl discriminant and $v_M^G(x)$ Arthur's weight function.

The tensor product gives an homomorphism of complex dual groups ($GL_n^\vee = GL_n(\mathbb{C})$)

$$H(\mathbb{C}) = G_n(\mathbb{C}) \times G_m(\mathbb{C}) \longrightarrow G_n(\mathbb{C}) \otimes G_m(\mathbb{C}) \hookrightarrow G(\mathbb{C})$$

Their L-groups are simply the direct products with the Galois or Weil group, so we have an L-homomorphism. Note that the map is not an embedding, as the kernel consists of diagonal elements isomorphic to (x, x^{-1}). By the Chevalley and Satake isomorphisms, the homomorphism induces a contravariant map of spherical Hecke algebras

$$\begin{array}{ccc} \mathbb{C}[G]^G & \xrightarrow{\sim} & \mathcal{H}(G) \\ \big\downarrow & & \big\downarrow t \\ \mathbb{C}[H]^H & \xrightarrow{\sim} & \mathcal{H}(H) \end{array}$$

and we will also identify $\mathcal{H}(GL_n \times GL_m)$ with $\mathcal{H}(GL_n) \otimes \mathcal{H}(GL_m)$. We will call t the tensor product map, in analogy with the base change map b.

A transfer conjecture in this context should be along the following lines:

Conjecture 1.2.1. *For all f in $\mathcal{H}(G(k_v))$ there exists $f_1 \otimes f_2$ in $\mathcal{H}(G_n(k_v)) \otimes \mathcal{H}(G_m(k_v))$ such that up to a transfer factor,*

$$\mathbf{O}_{\gamma_{12}}(f) = \Delta_H^G(\gamma_{12})\mathbf{O}_{\gamma_1}(f_1)\mathbf{O}_{\gamma_2}(f_2)$$

where $\Delta_H^G(\gamma_{12})$ is nonzero only if $(\gamma_1, \gamma_2) \leftrightarrow \gamma_{12}$.

Call such $f_1 \otimes f_2$ and f *associated*. Furthermore, the transfer should be compatible with tensor products, that is, we would like f and $t(f)$ to be associated. The transfer factor Δ_H^G is not well-defined at the moment, but one expects that it may be large in our case, as the image of H in G is rather small.

One might naively formulate a fundamental lemma for Lie algebras, in analogy with standard endoscopy, that is to say, the unit elements $1_{\mathfrak{g}(\mathcal{O})}$ and $1_{\mathfrak{g}_n(\mathcal{O})} \otimes 1_{\mathfrak{g}_m(\mathcal{O})}$ are associated. But unfortunately, by examining the proof of the fundamental lemma by Ngô, we will see that this last statement appears to be false. In particular, by the Support Theorem the bases of the associated Hitchin fibrations are not equal, thus one is not able to match perverse sheaves as desired.

2 Geometric objects

In this section fix C a smooth, geometrically connected, projective curve of genus g over a finite field $k = \mathbb{F}_q$ of large characteristic, and $\overline{C} = C \times_k \overline{k}$. Denote by F the function field of C, F_v the fraction field of the completed local ring at a closed point v in C, $\mathcal{O}_v$ its ring of integers, and k_v the residual field.

2.1 Local objects: the affine Springer fibre

A lattice L in k^n is a free rank n $\mathcal{O}_v$-module. $G(F)$ acts transitively on the set of lattices with stabilizer $G(\mathcal{O}_v)$. We say L is γ-stable if $\gamma(L) \subset L$; if $L = g\mathcal{O}^n$ then L is γ-stable if and only if $g^{-1}\gamma g \in \mathfrak{g}(\mathcal{O}_v)$. $G(F_v)/G(\mathcal{O}_v)$ is the affine Grassmannian parametrizing lattices in k^n. Let $\gamma = \epsilon(a)$. Define the set of γ-stable lattices

$$M^i_{v,a}(k) = \{\mathcal{O}_v\text{-lattices in } F_v^n : \gamma(L) \subset L \text{ and } \wedge^n L = t^i \mathcal{O}_v\},$$

where t is a uniformizer of $\mathcal{O}_v$. By the G action, it is in bijection with

$$\{g \in G(F_v)/G(\mathcal{O}_v) : \mathrm{Ad}(g)^{-1}(\gamma) \in \mathfrak{g}(\mathcal{O}_v) \text{ and } \mathrm{val}_{F_v}(\det g) = i\}.$$

The centralizer $I_\gamma(F_v)$ induces an action of $P_{v,a}(k)$ where

$$P_{v,a}(k) = J_a(F_v)/J_a(\mathcal{O}_v) \simeq I_\gamma(F_v)/I_\gamma(\mathcal{O}_v).$$

The *affine Springer fibre* is defined as

$$M_{v,a} = \coprod_i M^i_{v,a}$$

The following identity is the starting point of the theory.

Proposition 2.1.1. *Let γ be a regular semisimple element in $\mathfrak{g}(F_v)$. The cocharacter group X_* of $T = I_\gamma$ acts on $M_{v,a}(k)$ with no fixed points and $\mathbf{O}_\gamma(1_{\mathfrak{g}(\mathcal{O}_v)}) = |X_*(T)\backslash M_{v,a}(k)|$.*

Proof. We sketch the proof given by Chaudouard: We want to count γ-stable lattices in F_v^n modulo the action of T_{F_v}. Any element of T fixes γ, so the action preserves the fiber $M_{v,a}$. Restricting to the subgroup $X_*(T)$, the stabilizer of a lattice is a compact discrete subgroup hence a finite group, which must be trivial since $X_*(T)$ is a free $\mathbb{Z}$-module. So $X_*(T)$ acts freely. Now

$$\mathbf{O}_\gamma(1_{\mathfrak{g}(\mathcal{O}_v)}) = \int_{X_*(T)\backslash G_F} 1_{\mathfrak{g}(\mathcal{O}_v)}(x^{-1}\gamma x)dx = \sum_{X_*(T)\backslash G_F/G_{\mathcal{O}_v}} \int_{G_{\mathcal{O}_v}} 1_{\mathfrak{g}(\mathcal{O}_v)}(k^{-1}x^{-1}\gamma x k)dk$$

requiring $\mathrm{vol}(G_{\mathcal{O}_v}) = 1$,

$$= \sum_{X_*(T)\backslash G_F/G_{\mathcal{O}_v}} 1_{\mathfrak{g}(\mathcal{O}_v)}(x^{-1}\gamma x) = |X_*(T)\backslash M_{v,a}(k)|.$$

$\square$

2.2 Global objects: the Hitchin fibration

Now introduce the *Hitchin stack* associating to k-schemes the groupoid:

$$\mathcal{M}_{\mathrm{GL}_n} = \mathcal{M} : S \mapsto \begin{cases} E, & \text{rank } n \text{ vector bundles on } C \times S \text{ with} \\ \phi, & \text{a twisted endomorphism } E \to E \otimes_{\mathcal{O}_C} \mathcal{O}_C(D) \end{cases}$$

where D is an even, effective divisor on C, and (E, ϕ) is called a *Higgs bundle.*

One also has the *Picard stack*

$$\mathcal{P} : S \mapsto \Big\{ \ J_a\text{-torsors on } C \times S \text{ for every } a \text{ in } \mathcal{A}$$

where the scheme $\mathcal{A}$ is the *Hitchin base* $\mathcal{A}$ is the affine space of characteristic polynomials,

$$t^n - \mathrm{tr}(\phi)t^{n-1} + \cdots + (-1)^n \mathrm{tr}(\wedge^n \phi)$$

identified with the space of global sections of $C \times S$ with values in $\mathfrak{c}_D := \mathfrak{c} \otimes_{\mathcal{O}_C} \mathcal{O}_C(D)$

$$\mathcal{A} = \bigoplus_{1 \leq i \leq n} H^0(C, \mathcal{O}_C(iD))$$

The *Hitchin fibration* is the morphism $f : \mathcal{M} \to \mathcal{A}$ sending a pair

$$(E, \phi) \mapsto (\mathrm{tr}(\phi), \mathrm{tr}(\wedge^2 \phi), \ldots, \mathrm{tr}(\wedge^n \phi))$$

where $\mathrm{tr}(\wedge^i \phi)$ is the trace of the endomorphism $\wedge^i \phi : \wedge^i E \to \wedge^i E \otimes \mathcal{O}_C(iD)$.

The *Hitchin fibres* $\mathcal{M}_a := f^{-1}(a)$ of the map are the global analogue of the affine Springer fibres. Consider the total space of the line bundle $\mathcal{O}_C(D)$

$$\pi : \Sigma_D = \mathrm{Spec}(\bigoplus_{i=1}^{\infty} \mathcal{O}_C(-iD)t^i) \to C.$$

Given a $\overline{k}$-point a of $\mathcal{A}$, we have the characteristic polynomial $p_a : \Sigma_D \to \Sigma_D^n$

$$p_a(t) = t^n + a_1(v)t^{n-1} + \cdots + a_n(v) = 0, \qquad a_i \in H^0(C, \mathcal{O}_C(iD))$$

The *spectral curve* Y_a is the preimage under p_a of the zero section of Σ_D^n, tracing out a closed curve in Σ_D defined by $p_a(t) = 0$. It is an n-fold finite cover of C, generically étale Galois of Galois group $W = S_n$. The following characterization is known:

Proposition 2.2.1. *The spectral curve Y_a is reduced and connected. Its irreducible components are in one-to-one correspondence with the irreducible factors of the characteristic polynomial $p_a(t)$.*

The relation of the spectral curve is the correspondence of Hitchin and Beauville-Narasimhan-Ramanan:

Theorem 2.2.2. *If a is regular, the Hitchin fibre $\mathcal{M}_a$ is isomorphic to the moduli stack of coherent torsion-free sheaves $\mathcal{F}$ of generic rank 1 on Y_a with a trivialization of the stalk over ∞. As C is a curve this is the compactified Jacobian:*

$$\mathcal{M}_a \simeq \overline{\mathrm{Pic}^0(Y_a)}.$$

where $Pic^0(Y_a)$ is the neutral component of the Picard scheme.

More generally, there is an isomorphism of stacks over the regular locus $\mathcal{A}^{\mathrm{reg}}$,

$$\mathcal{M}^{\mathrm{reg}} = \mathcal{M} \times_{\mathcal{A}} \mathcal{A}^{\mathrm{reg}} \simeq \overline{\mathrm{Pic}}(Y/\mathcal{A}^{\mathrm{reg}}) = \coprod_i \overline{\mathrm{Pic}}^i(Y/\mathcal{A}^{\mathrm{reg}})$$

where $\overline{\mathrm{Pic}}(Y/\mathcal{A}^{\mathrm{reg}})$ is the compactified Picard stack, containing the usual Picard stack as an open substack. Each of the components $\overline{\mathrm{Pic}}^i$ is a finite-type algebraic stack over $\mathcal{A}^{\mathrm{reg}}$.

Next consider the C-group schemes G_n and G_m, and the product

$$H = G_n \times_C G_m.$$

We have the Hitchin morphisms $f_n : \mathcal{M}_n \to \mathcal{A}_n$ and $f_m : \mathcal{M}_m \to \mathcal{A}_m$, and the product

$$f_H : \mathcal{M}_H = \mathcal{M}_n \times_k \mathcal{M}_m \to \mathcal{A}_H = \mathcal{A}_n \times_k \mathcal{A}_m$$

together with morphisms $\nu : \mathcal{A}_H \to \mathcal{A}$ given by the resultant as before, where

$$\mathcal{A} = \bigoplus_{i=1}^{nm} H^0(C, \mathcal{O}_C(iD)),$$

and $\mathcal{M}_H \to \mathcal{M}$ sending the pair $((E_n, \phi_n), (E_m, \phi_m))$ to $(E_n \otimes E_m, \phi_n \otimes \phi_m)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M}_H \arrow[r] \arrow[d, "f_H"] & \mathcal{M} \arrow[d, "f"] \\
\mathcal{A}_H \arrow[r, "\nu"] & \mathcal{A}
\end{tikzcd}$$

The universal spectral curve $Y_H \to \mathcal{A}_H$ is defined as the disjoint sum of the relative curves

$$Y_H = Y_n \times_k \mathcal{A}_m \sqcup \mathcal{A}_n \times_k Y_m$$

where $Y_n \subset \mathcal{A}_n \times_k \Sigma$ and $Y_m \subset \mathcal{A}_m \times_k \Sigma$ are the universal spectral curves for G_n and G_m. It is the partial normalization of the sum

$$Y_H \to \mathcal{A}_H \times_{\mathcal{A}} Y = Y_n \times_k \mathcal{A}_m + \mathcal{A}_n \times_k Y_m.$$

The Picard stack over $\mathcal{A}_H$ is $P_H = \mathrm{Pic}_{Y_H/\mathcal{A}_H} = P_n \times_k P_m$.

Remark 2.2.3. *L-packets and substacks of $\mathcal{M}_{GL_n}$.* From the theory of L-packets, we call an L-parameter

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G(\mathbb{C})$$

stable if it does not factor through any endoscopic subgroup H of G. Stability is trivial for GL_{nm}, while a fortriori the tensor product $GL_n \otimes GL_m$ is not contained in any proper Levi, the non-elliptic endoscopic groups. Thus the L-parameter of $GL_n \otimes GL_m$ is necessarily stable, and we must decompose the perverse sheaf beyond endoscopy, so to speak.

In parallel, we find various substacks of the Hitchin stack associated to GL_n by imposing further conditions. For example, starting with

$$\mathcal{M}_{\mathrm{GL}_n} : S \mapsto \begin{cases} E, \text{ rank } n \text{ vector bundles on } C \times S \text{ with} \\ \phi, \text{ a twisted endomorphism } E \to E \otimes_{\mathcal{O}_C} \mathcal{O}_C(D) \end{cases}$$

we have also

$$\mathcal{M}_{\mathrm{SL}_n} : S \mapsto \begin{cases} E, \text{ rank } n \text{ vector bundles on } C \times S \text{ with trivialized determinant } \wedge^n E \simeq \mathcal{O}_C \\ \phi, \text{ a twisted endomorphism } E \to E \otimes_{\mathcal{O}_C} \mathcal{O}_C(D) \text{ with trace } \mathrm{tr}(\phi) = 0 \in \Gamma(C, \mathcal{O}_C(D)) \end{cases}$$

Similarly, the abelian varieties associated to each Hitchin stack is are related via their spectral curves.

A side question one may ask is: can these substacks give information about L-parameters?

2.3 Decomposition of perverse sheaves

First, we recall some results, mainly due to Ngô. We concentrate on the anisotropic locus of the Hitchin fibration, $\tilde{f}^{\mathrm{ell}} : \tilde{\mathcal{M}}^{\mathrm{ell}} \to \tilde{\mathcal{A}}^{\mathrm{ell}}$, where the tilde indicates we are working over a certain étale open subset. To ease notation, denote $M = \tilde{\mathcal{M}}^{\mathrm{ell}}$ and $A = \tilde{\mathcal{A}}^{\mathrm{ell}}$

The morphism $\tilde{f}^{\mathrm{ell}}$ is proper, so by Deligne, the complex of ℓ-adic sheaves $R\tilde{f}^{\mathrm{ell}}_* \overline{\mathbb{Q}}_l$ is pure. Then the decomposition theorem of Gabber and Beilinson-Bernstein-Deligne implies the following isomorphism in $D^b_c(A, \overline{\mathbb{Q}}_l)$,

$$R\tilde{f}^{\mathrm{ell}}_* \overline{\mathbb{Q}}_l = \bigoplus_{i \in \mathbb{Z}} {}^p H^i (R\tilde{f}^{\mathrm{ell}}_* \overline{\mathbb{Q}}_l)[-i]$$

where each perverse cohomology sheaf is semisimple, i.e., splits as a direct sum of intersection cohomology complexes associated with lisse irreducible sheaves on subschemes A,

$${}^p H^\bullet (R\tilde{f}^{\mathrm{ell}}_* \overline{\mathbb{Q}}_l[\dim(M)]) = \bigoplus_{a \in A} i_{a*} j_{a!*} \mathcal{F}^\bullet_a [\dim(a)]$$

where the sum is over Zariski points of A with the canonical inclusion $i_a : \overline{\{a\}} \hookrightarrow A$, and $\mathcal{F}^\bullet_a$ is a graded local system on a smooth open subset $j_a : U \hookrightarrow \overline{\{a\}}$.

Define the *socle* of a perverse sheaf the finite set of a in A such that $\mathcal{F}_a^\bullet$ is nonzero. Denote by A^{good} the subset of a in A such that $\mathrm{codim}_A(a) \geq \delta_a$ where δ_a is the sum of Serre invariants

$$\delta_a = \sum_{y \in |Y_a|} \mathrm{length}(\tilde{\hat{\mathcal{O}}}_{Y_a,y} / \hat{\mathcal{O}}_{Y_a,y}).$$

The theorem below is due to Ngô over the elliptic set, which Chaudouard-Laumon extended to outside the elliptic set by truncating the Hitchin stack. It is a consequence of Ngô's more general Support Theorem. We state it only in the simplest case of $G = GL_n$.

Theorem 2.3.1. *(Ngô) The socle of ${}^pH^\bullet(R\tilde{f}_*^{\mathrm{ell}}\overline{\mathbb{Q}}_l)$ over A^{good} contains a single element, namely the generic point. In other words, the support of a simple constituent ${}^pH^\bullet(R\tilde{f}_*^{\mathrm{ell}}\overline{\mathbb{Q}}_l)$ over A^{good} is A^{good} itself.*

Remark 2.3.2. In general, the action of $\mathcal{P}$ on $\mathcal{M}$ gives an action of the sheaf of abelian groups $\pi_0(\tilde{\mathcal{P}}^{\mathrm{ell}})$, a quotient of the constant sheaf, on the summands. Thus the cocharacter group X_* acts through π_0, so that a character of the finite abelian group descends to an endoscopic character:

$$X_*(T) \to \pi_0(\tilde{\mathcal{P}}^{\mathrm{ell}}) \xrightarrow{\kappa} \overline{\mathbb{Q}}_l^\times.$$

Consequently the perverse cohomology breaks into κ-isotypic pieces:

$${}^pH^n(\tilde{f}_*^{\mathrm{ell}}\overline{\mathbb{Q}}_l) = \bigoplus_{\kappa \in \check{T}} {}^pH^n(\tilde{f}_*^{\mathrm{ell}}\overline{\mathbb{Q}}_l)_\kappa$$

where the trivial character is defined to be the stable piece. Ngô and Laumon's work on the fundamental lemma shows that the non-trivial characters correspond to stable perverse cohomology attached to endoscopic groups. The construction is completely parallel to the original formulation of Langlands and Shelstad.

2.4 Tensor products

Let $\mathbf{A}^\bullet, \mathbf{B}^\bullet$ be complexes on X_1, X_2 respectively over S, and let $\mathrm{pr}_i : X_1 \times_S X_2 \to X_i$ be the projections onto to each factor $i = 1, 2$. The *external tensor product* is the usual tensor product of the pullbacks

$$\mathbf{A}^\bullet \overset{L}{\boxtimes}_S \mathbf{B}^\bullet := \mathrm{pr}_1^*\mathbf{A}^\bullet \overset{L}{\otimes}_S \mathrm{pr}_2^*\mathbf{A}^\bullet$$

There is a Künneth formula in the bounded derived category of constructible sheaves. Suppose we have two maps $f_1 : X_1 \to Y_1$ and $f_2 : X_1 \to Y_2$ over S inducing the map $f = f_1 \times_S f_2 : X_1 \times_S X_2 \to Y_1 \times_S Y_2$. Then given two complexes $\mathbf{A}^\bullet \in D_c^b(X_2)$ and $\mathbf{B}^\bullet \in D_c^b(X_2)$, the Künneth isomorphism is

$$Rf_!(\mathbf{A}^\bullet \overset{L}{\boxtimes}_S \mathbf{B}^\bullet) = Rf_{1!}(\mathbf{A}^\bullet) \overset{L}{\boxtimes}_S Rf_{2!}(\mathbf{B}^\bullet)$$

Note that when f is proper morphism, $f_! = f_*$, so we may decompose the constant sheaf $Rf_*\overline{\mathbb{Q}}_l$ according to Theorem 2.3.1 since f_1, f_2 proper implies $f_1 \times_S f_2$ is also.

Now let $\mathbf{A}^\bullet, \mathbf{B}^\bullet$ be graded local systems on smooth open subsets $j_i : U_i \hookrightarrow A_i$ whose intermediate extensions are constituents of ${}^pH^\bullet(R\tilde{f}_{i*}^{\mathrm{ell}}\overline{\mathbb{Q}}_l[\dim(M)])$ for $i = n, m$. Their support is necessarily A_i. Next consider the complex over A_H,

$$K_H := Rf_{1*}(\mathbf{A}^\bullet) \overset{L}{\boxtimes}_S Rf_{2*}(\mathbf{B}^\bullet).$$

We want to study its direct image $\nu_*(K_H)$ over A. The first step is to determine its support.

Let's recall Ngô's Support Theorem in full:

Theorem 2.4.1. [N, 7.2.1] *Let S be a geometrically irreducible finite-type k-scheme. Let $f : M \to S$ and $g : P \to S$ be a δ-regular abelian fibration of relative dimension d with the total space M smooth over k. Let K be a simple perverse sheaf occurring in $f_*\overline{\mathbb{Q}}_l$ with support Z. Then there exists an open subset U of $S \otimes_k \bar{k}$ such that $U \cap Z \neq \varnothing$ and a nontrivial local system L on $U \cap Z$ such that L is a direct factor of $R^{2d}f_*\overline{\mathbb{Q}}_l|_U$. In particular, if the geometric fibre is irreducible then $Z = S \otimes_k \bar{k}$.*

When $G = GL_n$, this implies

Theorem 2.4.2. [N, 7.8.3] *Let K be a geometrically irreducible perverse factor of ${}^pH^n(\tilde{f}^{\mathrm{ell}}\overline{\mathbb{Q}}_l)$ with support Z. Suppose $Z \cap A^{\mathrm{good}} \neq \varnothing$. Then $Z = A^{\mathrm{ell}}$.*

Indeed, one sees that $\nu(A_H)$ intersects A^{ell} nontrivially:

Lemma 2.4.3. *Given $\nu : A_H \to A$ as above, $\nu(A_H) \cap A^{\mathrm{ell}} \neq \varnothing$.*

Proof. It is enough to produce an irreducible polynomial $\nu(a_1, a_2)$. Choose a_i in A_i^{ell} such that the roots of the irreducible polynomials p_{a_i} defined by a_i are a normal basis for finite Galois extensions of F of degree n and m. By construction the roots of $p_{\nu(a_1,a_2)}$ are the product of the roots of p_{a_1} and p_{a_2}, hence it is a minimal polynomial of an extension of degree mn. □

But this implies that if $\nu_*(K_H)$ occurs in the decomposition of $f_*\overline{\mathbb{Q}}_l$ over A^{ell}, then its support would be all of A^{ell}. But we know that the image $\nu(A_H)$ is strictly contained in A, contradicting our naive guess.

So we introduce a modification: the image $\nu(\mathcal{M}_H)$ is the substack of $\mathcal{M}_G$ parametrizing rank n and m vector bundles $E_n \otimes E_m$ and endomorphisms $\phi_n \otimes \phi_m$ with characteristic polynomial of the form $p_{A\otimes B}$ as in §1.1. Define the skyscraper sheaf $\bar{\mathbb{Q}}_{\ell,H}$ on $\mathcal{M}_G$ nonzero over $\nu(\mathcal{M}_H)$.

Definition 2.4.4. Recall that a $\bar{\mathbb{Q}}_\ell$-sheaf F on X is *punctually pure of weight w* if the eigenvalues of Frob^n acting on the stalks F_x have absolute value $q^{nw/2}$ for every x in $X(\mathbb{F}_{q^n})$. Furthermore, a complex K in $D_c^b(X, \bar{\mathbb{Q}}_\ell)$ is *pure of weight w* if the cohomology sheaves $H^i(K)$ and $H^i(K^\vee)$ are punctually pure of weights $\leq w + i$.

Over the stack $\mathcal{M}_G$ the sheaf $\bar{\mathbb{Q}}_{\ell,H}$ is a mixed complex. Denote by $\mathcal{M}_G^{\text{ani}}$ the anisotropic locus

Now by Deligne, the proper pushforward of a pure complex is pure, and once again the decomposition theorem applies. Then we need to look for the supports of the simple constituents, which leads us to Ngô's Support Theorem. How can we adapt this?

3 Analytic method

Assuming the fundamental lemma above, a trace identity should follow along the lines of Labesse's work. Having the trace identity then the theorem of Moeglin and Waldspurger should be necessary:

Theorem 3.0.1. (Moeglin-Waldspurger) *The irreducible representations of $GL_n(\mathbb{A})$ occurring in the discrete $L^2(GL_n(F)\backslash GL_n(\mathbb{A})^1)$ spectrum have multiplicity one. They are parametrized by pairs (k, σ) where σ is an irreducible unitary cuspidal automorphic representation of $GL_k(\mathbb{A})$, with $k|n$. If P is the standard parabolic of type $(k, \ldots, k)$ in GL_n, and ρ_σ is the nontermpered representation of $M_P(\mathbb{A}) \simeq GL_k(\mathbb{A})^p$*

$$(\sigma \otimes \cdots \otimes \sigma)\delta_P^{\frac{1}{2}} : m \mapsto (\sigma(m_1)|\det m_1|^{\frac{p-1}{2}}) \otimes \cdots \otimes (\sigma(m_1)|\det m_1|^{-\frac{p-1}{2}}),$$

then π is the unique irreducible quotient of the induced representation $\mathrm{Ind}_P^{GL_n}(\rho_\sigma)$.

And also the theorem of Jacquet and Shalika:

Theorem 3.0.2. (Jacquet-Shalika) *Let π_1 and π_2 be cuspidal unitary automorphic representations of GL_n, S a finite set of places containing the archimedean ones and outside which the π_i are unramified. Then the partial L-function $L^S(s, \pi_1 \times \tilde{\pi}_2)$ is regular nonzero for $\mathrm{Re}(s) > 1$, and at $s = 1$ has a pole if $\pi_1 \simeq \pi_2$, regular nonzero otherwise.*

References

[1] Chaodouard, G *Geometry of the fundamental lemma.*

[2] Laumon, G., Ngô B.C., *Le lemme fondamental pour les groupes unitaires.*

[3] Ngô B.C., *Le lemme fondamental pour les algèbres de Lie.*