BILINEAR FORMS WITH KLOOSTERMAN SUMS

1. Introduction

For K an arithmetic function and $\tilde{\alpha} = (\alpha_m)_1^\infty$, $\tilde{\beta} = (\beta)_1^\infty$ complex coefficients, it is often useful to estimate bilinear forms of the shape

$$B(K, \tilde{\alpha}, \tilde{\beta}) = \sum_m \sum_n \alpha_m \beta_n K(mn).$$

With applications to modular forms in mind, we restrict our attention to the situation in which K is a Kloosterman or hyper-Kloosterman sum, i.e., for some $k \geq 2$ we have

$$\begin{aligned} K = \mathrm{Kl}_k(\cdot, q) : (\mathbf{Z}/q\mathbf{Z})^\times &\to \mathbf{C} \\ n &\mapsto q^{\frac{1-k}{2}} \sum_{\substack{x_1, \ldots, x_k \in (\mathbf{Z}/q\mathbf{Z})^\times \\ x_1 \ldots x_k = n}} e_q(x_1 + \cdots + x_k) \end{aligned}$$

We can extend K to an arithmetic function by 0, or in other controlled ways. Since the prime q shall be somewhat large compared to the supports of $\tilde{\alpha}$ and $\tilde{\beta}$, the precise nature of this extension does not affect our results. Our coefficients $\tilde{\alpha}$ shall be supported on $[M] := \{1, \ldots, M\}$, while our coefficients $\tilde{\beta}$ shall be supported on an interval $N \subset [1, q-1]$ of length N.

Since $||K||_\infty \ll 1$ (as a well-known consequence of Deligne's work), we can use Cauchy or Hölder to bound $B(K, \tilde{\alpha}, \tilde{\beta})$ *trivially*, for instance,

$$B(K, \tilde{\alpha}, \tilde{\beta}) \ll_n ||\tilde{\alpha}||_2 ||\tilde{\beta}||_2 (MN)^{1/2} (q^{-1/4} + M^{-1/2} + N^{-1/2} q^{1/4} \log q),$$

an estimate that is nontrivial in the ranges

$$M \geq q^\delta, N \geq q^{1/2+\delta}$$

for some $\delta > 0$, for instance.

A fundamental challenge, when dealing with incomplete character sums, is to go beyond the Pòlya-Vinogradov range.)For Dirichlet Characters, Burgess bounds are the archetype [...].) This was achieved in the present context by Kowalski, Michel, and Sawin [**?**].

Theorem 1.1 ([**?**]). *Let q be a prime, and let $M, N \in \mathbf{R}$ satisfy*

$$1 \leq M \leq N q^{1/4}, \qquad q^{1/4} < MN < q^{5/4}.$$

Then for any $\epsilon > 0$ we have

$$B(K, \tilde{\alpha}, \tilde{\beta}) \ll_{n,\epsilon} q^\epsilon ||\tilde{\alpha}||_2 ||\tilde{\beta}||_2 (MN)^{1/2} (M^{-1/2} + (MN)^{-3/16} q^{11/64}).$$

This is nontrivial when $M = N \geq q^{11/24} + \delta$, for instance. We offer the following bound, which goes further beyond the Pòlya-Vinogradov range:

Theorem 1.2. *Assume*

$$1 \leq M^2 \leq Nq, \qquad q^{7/8} \leq MN \leq \frac{q^2}{64}.$$

Then

$$B(K, \tilde{\alpha}, \tilde{\beta}) \ll_{n,\epsilon} q^{\epsilon} ||\tilde{\alpha}||_2 ||\tilde{\beta}||_2 (MN)^{1/2} (M^{-1/2} + (q^7 (MN)^{-8})^{1/72}).$$

This beats the trivial bound when $M = N \geq q^{\frac{7}{16}+\delta}$, for instance.

In applications, it is often beneficial to have specific bounds tailored to the scenario in which $\tilde{\beta} = 1_{\mathcal{N}}$. This is the 'Type I' scenario arising in the Vaughan [**?**] and Heath-Brown [**?**] identities, the more general situation addressed in Theorem 1.1 is known as 'Type II'. Kowalski, Michel, and Sawin obtained the following Type I estimate:

Theorem 1.3 ([**?**])**.** *Assume* $||\tilde{\alpha}||_\infty \leq 1$, *and that*

$$1 \leq M \leq N^2, N < q, MN < q^{\frac{3}{2}}.$$

Then

$$B(K, \tilde{\alpha}, 1_{\mathcal{N}}) \ll q^{\epsilon} ||\tilde{\alpha}||_1^{1/2} ||\tilde{\alpha}||_2^{1/2} M^{1/4} N \Big(\frac{M^2 N^5}{q^3} \Big)^{1/12}.$$

Note that Cauchy gives $||\tilde{\alpha}||_1 \leq M^{1/2} ||\tilde{\alpha}||_2$, so a trivial bound is

$$B(K, \tilde{\alpha}, 1_{\mathcal{N}}) \ll N ||\tilde{\alpha}||_1 \ll ||\tilde{\alpha}||_1^{1/2} ||\tilde{\alpha}||_2^{1/2} M^{1/4} N.$$

Theorem 1.3 beats this when $M = N \geq q^{3/7+\delta}$, for instance.

Theorem 1.4. *Assume that* $||\tilde{\alpha}||_\infty \leq 1$ *and*

$$1 \leq M \leq N^3, MN \leq q.$$

Then

$$B(K, \tilde{\alpha}, 1_{\mathcal{N}}) \ll q^{\epsilon} ||\tilde{\alpha}||_1^{2/3} ||\tilde{\alpha}||_2^{1/3} M^{1/6} N \Big(\frac{q^4}{M^3 N^7} \Big)^{1/24}.$$

This defeats the trivial estimate

$$B(K, \tilde{\alpha}, 1_{\mathcal{N}}) \ll N ||\tilde{\alpha}||_1 \ll ||\tilde{\alpha}||_1^{2/3} ||\tilde{\alpha}||_2^{1/3} M^{1/6} N.$$

as soon as $M = N \geq q^{2/5+\delta}$, say.

2. Proof of Theorem 1.4

To prove Theorem 1.4, we begin as in [**?**, §2] with the '$+ab$-shifting' trick. Given parameters $A, B \geq 1$ such that

$$AB \leq N, AM < q,$$

we have

$$B(K, \tilde{\alpha}, \mathcal{N}) \ll \frac{q^{\epsilon}}{AB} \sum_{\substack{r \bmod q \\ s \leq 2AM}} \nu(r,s) \mu(r,s)$$

where

$$\nu(r,s) = \dots$$

(note the $\mathcal{N}$ here should be $\mathcal{N}'$, an interval of length $\leq 2N$) and

$$\mu(r,s) = \Big| \sum_{B < b \leq 2B} \eta_B K(s(r+b)) \Big|.$$

For ν, we have the moment estimates

$$||\nu||_1 \ll AN||\tilde{\alpha}||_1$$

and

$$||\nu||_2^2 \ll q^\epsilon AN||\tilde{\alpha}||_2^2$$

from [**?**].

Now we apply Hölder's inequality with exponent 6:

$$\begin{aligned}\sum_{\substack{r \bmod q \\ s \le 2AM}} \nu(r,s)\mu(r,s) &= ||\nu\mu||_1 \\ &\le ||\nu^{2/3}||_{3/2}||\nu^{1/3}||_6||\mu||_6 \\ &\ll (AN||\tilde{\alpha}||_1)^{2/3}(q^\epsilon AN||\tilde{\alpha}||_2^2)^{1/6}||\mu||_6\end{aligned}$$

We adapt the standard notational convention that ϵ denotes an arbitrarily small positive number, whose value may differ between instances. After a small amount of bookkeeping, we now have

$$B(K,\tilde{\alpha},\mathcal{N}) \ll \frac{q^\epsilon}{AB}(AN)^{\frac{5}{6}}||\tilde{\alpha}||_1^{2/3}||\tilde{\alpha}||_2^{1/3}||\mu||_6. \tag{1}$$

By the triangle inequality, we have

$$||\mu||_6^6 \le \sum_{\tilde{b}\in\mathcal{B}} |S(K,\tilde{b};2AM)|$$

where

$$\mathcal{B} = \{\tilde{b} \in \mathbf{Z}^6 : B < b_i \le 2B, 1 \le i \le 6\}$$

and

$$S(K,\tilde{b},2AM) = \sum_{r \bmod q}\prod_{i=1}^{3} K(s(r+b_i))\overline{K}(s(r+b_{i+3})).$$

Here $\overline{K}(x) = \overline{K(x)}$.

Definition 2.1. *Let $\mathcal{V}^\Delta$ be the affine variety of sextuples*

$$\tilde{b} = (b_1,\ldots,b_6) \in \mathbf{A}^6_{\mathbf{F}_q}$$

defined by the conditions

1. *If k is even, then for any $i \in \{1,\ldots,6\}$ the cardinality $\#\{j : b_j = b_i\}$ is even.*
2. *If k is odd and not equal to 3, then $\{\{b_1,b_2,b_3\}\} = \{\{b_4,b_5,b_6\}\}$ is an equality of multisets.*
3. *If $k = 3$, then either $\{\{b_1,b_2,b_3\}\} = \{\{b_4,b_5,b_6\}\}$ or $\tilde{b} = (b,b,b,b',b',b')$ for some b,b'.*

The role of the 'diagonal set' is played by

$$\mathcal{B}^\Delta = \{\tilde{b} \in \mathcal{B} : \tilde{b} \bmod q \in \mathcal{V}^\Delta\}.$$

Observe that the contribution from the vectors $\tilde{b} \in \mathcal{B}^\Delta$ to $||\mu||_6^6$ satisfies

$$\sum_{b\in\mathcal{B}^\Delta} |S(K,\tilde{b};2AM)| \ll qAB^3M := x_1.$$

For $\tilde{b} \notin \mathcal{B}^{\Delta}$, we can exploit averaging over r:

Lemma 2.2. *For $b \in \mathcal{B}\backslash\mathcal{B}^{\Delta}$ and $s \in \mathbf{F}_q^{\times}$, we have*

$$\sum_{r \ mod \ q} \prod_{i=1}^{3} K(s(r+b_i))\overline{K}(s(r+b_{i+3})) \ll q^{1/2}.$$

In particular, for any $\mathcal{B}' \subset \mathcal{B}\backslash\mathcal{B}^{\Delta}$ we have

$$\sum_{\tilde{b} \in \mathcal{B}'} |S(K, \tilde{b}, 2AM)| \ll AMq^{1/2}|\mathcal{B}'|.$$

We refer to §3 for the proof. Generically we'll need to save more than $q^{1/2}$.

An application of the Plancherel formula—this is the Pòlya-Vinogradov method from §4 of our course notes—yields

$$S(K, \tilde{b}; 2AM) \ll (\log q) \max_{\lambda \bmod q} |\hat{S}(K, \tilde{b}, \lambda)|$$

where

$$\hat{S}(K, \tilde{b}, \lambda) = \sum_{r \bmod q} R(K, r, \lambda, \tilde{b})$$

with

$$\begin{aligned} R(K, r, \lambda, \tilde{b}) &= R(K, r, \lambda, \tilde{b}) \\ &= \sum_{s \bmod q}^{\times} e_q(\lambda s) \prod_{i=1}^{3} K(s(r+b_i))\overline{K}(s(r+b_{i+3})). \end{aligned}$$

By following the proof of [**?**, Theorem 2.3], we obtain the following generic estimate.

Theorem 2.3. *There exists a codimension one subvariety $\mathcal{V}^{\text{bad}} \subset \mathbf{A}^6_{\mathbf{F}_q}$ containing $\mathcal{V}^{\Delta}$, with degree bounded independently of q, such that if $\lambda \in \mathbf{F}_q$ and $\tilde{b} \notin \mathcal{V}^{\text{bad}}(\mathbf{F}_q)$ then $\hat{S}(K, \tilde{b}, \lambda) \ll q$ and therefore $S(K, \tilde{b}, 2AM) \ll q \log q$.*

This uses the full power of Deligne-Katz [**?**], but an improvement could still be sought on the codimension.

Write

$$\mathcal{B}^{\text{bad}} = \{\tilde{b} \in \mathcal{B} : \tilde{b} \bmod q \in \mathcal{V}^{\text{bad}}(\mathbf{F}_q)\}$$

and

$$\mathcal{B}^{\text{gen}} = \mathcal{B}\backslash\mathcal{B}^{\text{bad}}.$$

By Schwartz-Zippel and uniformity of the degree bound, we have $\#\mathcal{B}^{\text{bad}} \leq (\deg \mathcal{V}^{\text{bad}})B^5 \ll B^5$. Thus by Lemma 2.2 we have

$$\sum_{\tilde{b} \in \mathcal{B}^{\text{bad}} \backslash \mathcal{B}^{\Delta}} |S(K, \tilde{b}; 2AM)| \ll q^{1/2}AB^5M := x_2.$$

By Theorem 2.3 we have

$$\sum_{\tilde{b} \in \mathcal{B}^{\text{gen}}} |S(K, \tilde{b}; 2AM)| \ll (\log q)qB^6 := (\log q)x_3.$$

Thus

$$||\mu||_6^6 \ll (x_1 + x_2 + x_3) \log q$$

where $x_1 = qAB^3M, x_2 = q^{1/2}AB^5M, x_3 = qB^6$.

Choosing

$$A = M^{-1/4}N^{3/4}, \quad B = M^{1/4}N^{1/4}$$

ensures that $AB = N$ and $x_1 = x_3$.

We note that the hypotheses of our theorem ensure that

$$A \geq 1, \quad AM < q$$

as our parameters are acceptable. Moreover, the hypothesis $MN \leq q$ ensures that $x_2 \leq x_3 = q(MN)^{3/2}$.

Now from (1) we have

$$\begin{aligned} B(K,\tilde{\alpha},\mathcal{N}) &\ll \frac{q^{\epsilon}}{N}(AN)^{5/6}||\tilde{\alpha}||_1^{2/3}||\tilde{\alpha}||_2^{1/3}q^{1/6}(MN)^{1/4} \\ &= \frac{q^{\epsilon+1/6}}{N}N^{5/6}(\frac{N^{15}}{M^5})^{1/24}||\tilde{\alpha}||_1^{2/3}||\tilde{\alpha}||_2^{1/3}(MN)^{1/4} \\ &= q^{\epsilon+1/6}M^{1/24}N^{17/24}||\tilde{\alpha}||_1^{2/3}||\tilde{\alpha}||_2^{1/3} \\ &= q^{\epsilon}M^{1/6}N||\tilde{\alpha}||_1^{2/3}||\tilde{\alpha}||_2^{1/3}(\frac{q^4}{M^3N^7})^{1/24} \end{aligned}$$

We use a similar strategy to prove Theorem 1.2. This time Cauchy-Scwhartz, the +ab-shifiting trick, and Hölder-6 give

$$B(K,\tilde{\alpha},\tilde{\beta})^2 \ll ||\tilde{\alpha}||_2^2||\tilde{\beta}||_2^2(N + \frac{q^{\epsilon}}{AB}M^{2/3}(AN)^{5/6}||\mu'||_6),$$

where

$$||\mu||_6^6 = \sum_{\tilde{b}\in\mathcal{B}} |S^{\neq}(K,\tilde{b};2AM)|.$$

Here

$$S^{\neq}(K,\tilde{b};2AM) = \sum_{\substack{r \bmod q \\ s_2,s_2\leq 2AM \\ s1\neq s_2 \bmod q}} \prod_{i=1}^{3} K(s_1(r+b_i))\overline{K}(s_2(r+b_i))\overline{K}(s_1(r+b_{i+3}))K(s_2(r+b_{i+3})).$$

In §3, we shall also confirm the following analogue of Lemma 2.2:

Lemma 2.4. *For any subset $\mathcal{B}' \subset \mathcal{B}\backslash\mathcal{B}^{\Delta}$ we have*

$$\sum_{\tilde{b}\in\mathcal{B}'} |S^{\neq}(K,\tilde{b},2AM)| \ll (AM)^2q^{1/2}|\mathcal{B}'|.$$

For $\mathcal{B}^{\Delta}$, we have the trivial bound

$$\sum_{\tilde{b}\in\mathcal{B}'} |S^{\neq}(K,\tilde{b},2AM)| \ll qA^2B^3M^2 := y_1 \tag{2}$$

We WHAT the condition $s_1 \neq s_2 \bmod q$ by the indicator function expression,

$$1 - \frac{1}{q}\sum_{\lambda \bmod q} e_q(\lambda(s_1 - s_2)).$$

The Pólya-Vinogradov method then gives

$$S^{\neq}(K,\tilde{b},2AM) \ll (\log q)^2 + (\log q)^2 \max_{\substack{\lambda_1,\lambda_2 \bmod q \\ \lambda_1\neq\lambda_2}} |\hat{S}(K,\tilde{b},\lambda_1,\lambda_2)|.$$

Here

$$\hat{S}(K,\tilde{b},\lambda_1,\lambda_2)=\zeta(\lambda_1,\lambda_2,\tilde{b})-\frac{1}{q}\sum_{\lambda \bmod q}\zeta(\lambda_1+\lambda,\lambda_2+\lambda,\tilde{b}),$$

where

$$\zeta(\lambda_1,\lambda_2,\tilde{b})=\sum_{r \bmod q}R(r,\lambda_1,\tilde{b})\overline{R(r,\lambda_2,\tilde{b})}.$$

Mimicking the proof of [KMS, Theorem 2.5] gives

Theorem 2.5. *There exists a codimension one subvariety $\mathcal{V}^{bad}\subset \mathbf{A}^6_{\mathbf{F}_q}$ containing $\mathcal{V}^{\Delta}$, with degree bounded independently of q, such that for every $\tilde{b}\notin\mathcal{V}^{bad}(\mathbf{F}_q)$ and every distinct $\lambda_1,\lambda_2\in\mathbf{F}_q$, we have*

$$\hat{S}(K,\lambda_1,\lambda_2,\tilde{b})\ll q^{3/2}.$$

In fact, $\mathcal{V}^{\rm bad}$ is the same in Theorem 2.3 and Theorem 2.5. Using Lemma 2.4 for $\tilde{b}\in\mathcal{B}^{\rm bad}\backslash\mathcal{B}^{\Delta}$ and Theorem 2.5 for $\mathcal{B}^{\rm gen}$ and (2) for $\mathcal{B}^{\Delta}$ gives

$$||\mu'||_6^6\ll(\log q)^2(y_1+y_2+y_3)$$

where

$$y_1=qA^2B^3M^2,\quad y_2=q^{1/2}A^2B^5M^2,\quad y_3=q^{3/2}B^6.$$

Choosing

$$A=q^{1/3}M^{-2/3}N^{1/3},\quad B=q^{-1/3}M^{2/3}N^{2/3},$$

we have

$$AB=N,\quad y_2=y_3.$$

Moreover, the hypothesis $MN\geq q^{7/8}$ implies that $y_1\leq y_3=q^{-1/2}M^4N^4$. Now

$$\begin{aligned}\frac{B(K,\tilde{\alpha},\tilde{\beta})}{||\tilde{\alpha}||_2||\tilde{\beta}||_2}&\ll\sqrt{N}+q^{\epsilon}\Big(\frac{M^{2/3}(AN)^{5/6}}{AB}\Big)^{1/2}\Big(\frac{M^4N^4}{q^{1/2}}\Big)^{1/12}\\&=\sqrt{N}+q^{\epsilon-1/24}M^{1/3}(qM^{-2}N^4)^{5/36}M^{1/3}N^{1/6}\\&=\sqrt{N}+q^{\epsilon+7/72}M^{7/18}N^{7/18}\\&=\sqrt{N}+q^{\epsilon}(MN)^{1/2}(q^7(MN)^{-8})^{1/72}.\end{aligned}$$

Thus

$$B(K,\tilde{\alpha},\tilde{\beta})\ll q^{\epsilon}||\tilde{\alpha}||_2||\tilde{\beta}||_2(MN)^{1/2}(M^{-1/2}+q^7(MN)^{-8})^{1/72}.$$

3. Proof of lemmas

Proof of Lemma 2.2. We appeal directly to [**?**, Corollary 1.6]. The relevant vector is

$$\tilde{\gamma}=(\gamma_{s,1},\dots,\gamma_{s,6}),$$

where

$$\gamma_{s,i}=\begin{pmatrix}s & sb_i\\ 0 & 1\end{pmatrix},$$

for $i=1,\dots,6$.

- Case 1: k even. If $\tilde{b}\notin\mathcal{B}^{\Delta}$ then there exists an i such that $\#\{j:b_j=b_i\}$ is odd.

- Case 2: $k > 3$ odd. Here $\tilde{\sigma} = (1, 1, 1, c, c, c)$, where c denotes complex conjugation. If $\tilde{b} \notin \mathcal{B}^{\Delta}$, then
$$\{\{b_1, b_2, b_2\}\} \neq \{\{b_4, b_5, b_6\}\}.$$
In particular, there exists an i such that
$$\#\{j : b_i = b_j, j \geq 3\} \neq \#\{j : b_i = b_j, j \geq 4\}.$$
As known from [**?**, Remark 1.9], the special involution is
$$\xi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Since $q \neq 2$ and $s \in \mathbf{F}_q^{\times}$, we can never have $\xi\gamma_i = \gamma_j$. The conditions (2) and (3) of [**?**, Definition 1.3] are thus equivalent.
- Case 3: $k = 3$ This is almost the same as Case 2. The only thing that could go wrong is if $\tilde{\beta} = (b, b, b, b', b', b')$, but we have explicitly excluded this.

□

Proof of Lemma 2.4. We have
$$\tilde{\gamma} = (\gamma_{s_1,1}, \ldots, \gamma_{s_1,6}, \gamma_{s_2,1}, \ldots, \gamma_{s_2,6}$$
and
$$\tilde{\sigma} = (1, 1, 1, c, c, c, c, c, c, 1, 1, 1).$$
Recall that $s_1 \not\equiv s_2 \bmod q$.

- Case 1: k even. If $\tilde{b} \notin \mathcal{B}^{\Delta}$ then there exists an i such that $\#\{j : b_j = b_i\}$ is odd. Thus $\#\{j \leq 12 : \gamma_j = \gamma_i\}$ is odd; here $\gamma_i = \gamma_{s_1,i}$, since $i < b$.
- Case 2: $k > 3$ odd. If $\tilde{b} \notin \mathcal{B}^{\Delta}$ then
$$\#\{j : b_i = b_j, j \geq 3\} \neq \#\{j : b_i = b_j, j \geq 4\},$$
so there exists $i \leq 6$ such that
$$\#\{j \leq 3 : b_j = b_i\} \neq \#\{j \geq 4 : b_j = b_i\}.$$
and note that $\gamma_j \neq \gamma_i$ for $j > 6$ (as $s_1 \not\equiv s_2 \bmod q$). Since $k > 3$, this also means that the two expressions are incongruent modulo k.

  Also, k-normality of $(\tilde{\gamma}, \tilde{\sigma})$ is the same with or without respect to the special involution ξ, since we can never have $\xi\gamma_i = \gamma_j$. To see this, note that if $\xi\gamma_i = \gamma_j$ then
$$q|2s_i \text{ or } q|(s_1 + s_2),$$
for some $i = 1, 2$, both of which are impossible since $q \neq 2, s_i \in \mathbf{F}_q^{\times}$, and
$$4AM = 4(qMN)^{1/3} \leq q \Leftrightarrow 64MN \leq q^2$$
the latter given by our hypothesis.
- Case 3: $k = 3$. Again this is basically the same as Case 2, since we've explicitly forbidden vectors $\tilde{b}$ of the shape $(b, b, b, b'b'b')$.

□