1. Introduction

For \( K \) an arithmetic function and \( \tilde{\alpha} = (\alpha_m)_1^\infty, \tilde{\beta} = (\beta)_1^\infty \) complex coefficients, it is often useful to estimate bilinear forms of the shape
\[
B(K, \tilde{\alpha}, \tilde{\beta}) = \sum_m \sum_n \alpha_m \beta_n K(mn).
\]

With applications to modular forms in mind, we restrict our attention to the situation in which \( K \) is a Kloosterman or hyper-Kloosterman sum, i.e., for some \( k \geq 2 \) we have
\[
K = Kl_k(\cdot, q) : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}, n \mapsto q^{1-k/2} \sum_{x_1, \ldots, x_k \in (\mathbb{Z}/q\mathbb{Z})^*} e_q(x_1 + \cdots + x_k)
\]

We can extend \( K \) to an arithmetic function by \( 0 \), or in other controlled ways. Since the prime \( q \) shall be somewhat large compared to the supports of \( \tilde{\alpha} \) and \( \tilde{\beta} \), the precise nature of this extension does not affect our results. Our coefficients \( \tilde{\alpha} \) shall be supported on \( [1, M] := \{1, \ldots, M\} \), while our coefficients \( \tilde{\beta} \) shall be supported on an interval \( N \subset [1, q-1] \) of length \( N \).

Since \( ||K||_\infty \ll 1 \) (as a well-known consequence of Deligne’s work), we can use Cauchy or Hölder to bound \( B(K, \tilde{\alpha}, \tilde{\beta}) \) trivially, for instance,
\[
B(K, \tilde{\alpha}, \tilde{\beta}) \ll_n ||\tilde{\alpha}||_2 ||\tilde{\beta}||_2 (MN)^{1/2}(q^{-1/4} + M^{-1/2} + N^{-1/2}q^{1/4}\log q),
\]

an estimate that is nontrivial in the ranges
\[
M \geq q^\delta, N \geq q^{1/2+\delta}
\]

for some \( \delta > 0 \), for instance.

A fundamental challenge, when dealing with incomplete character sums, is to go beyond the Pólya-Vinogradov range. For Dirichlet Characters, Burgess bounds are the archetype \([\ldots]\). This was achieved in the present context by Kowalski, Michel, and Sawin \([?]\).

**Theorem 1.1** \((?[?])\). Let \( q \) be a prime, and let \( M, N \in \mathbb{R} \) satisfy
\[
1 \leq M \leq Nq^{1/4}, \quad q^{1/4} < MN < q^{5/4}.
\]

Then for any \( \epsilon > 0 \) we have
\[
B(K, \tilde{\alpha}, \tilde{\beta}) \ll_n, \epsilon q^{\epsilon} ||\tilde{\alpha}||_2 ||\tilde{\beta}||_2 (MN)^{1/2}(M^{-1/2} + (MN)^{-3/16}q^{11/64}).
\]

This is nontrivial when \( M = N \geq q^{11/24+\delta} \), for instance. We offer the following bound, which goes further beyond the Pólya-Vinogradov range:
Theorem 1.2. Assume
\[ 1 \leq M^2 \leq Nq, \quad q^{7/8} \leq MN \leq \frac{q^2}{64}. \]
Then
\[ B(K, \tilde{\alpha}, \tilde{\beta}) \ll n, \epsilon \left( ||\tilde{\alpha}||_2 ||\tilde{\beta}||_2 (MN)^{1/2} (M^{-1/2} + (q^7 MN^{-8})^{1/72}). \right) \]

This beats the trivial bound when \( M = N \geq q^{5/8} + \delta \), for instance.

In applications, it is often beneficial to have specific bounds tailored to the scenario in which \( \tilde{\beta} = 1_N \). This is the ‘Type I’ scenario arising in the Vaughan [?] and Heath-Brown [?] identities, the more general situation addressed in Theorem 1.1 is known as ‘Type II’. Kowalski, Michel, and Sawin obtained the following Type I estimate:

Theorem 1.3 ([?]). Assume \( ||\tilde{\alpha}||_\infty \leq 1 \), and that
\[ 1 \leq M \leq N^2, \quad N < q, \quad MN < q^2. \]
Then
\[ B(K, \tilde{\alpha}, 1_N) \ll q^{\epsilon} ||\tilde{\alpha}||_1^{1/2} ||\tilde{\alpha}||_2^{1/2} M^{1/4} N^2 \left( \frac{M^2 N^5}{q^3} \right)^{1/12}. \]

Note that Cauchy gives \( ||\tilde{\alpha}||_1 \leq M^{1/2} ||\tilde{\alpha}||_2 \), so a trivial bound is
\[ B(K, \tilde{\alpha}, 1_N) \ll N ||\tilde{\alpha}||_1 \ll ||\tilde{\alpha}||_1^{1/2} ||\tilde{\alpha}||_2^{1/2} M^{1/4} N. \]

Theorem 1.3 beats this when \( M = N \geq q^{3/7 + \delta} \), for instance.

Theorem 1.4. Assume that \( ||\tilde{\alpha}||_\infty \leq 1 \) and
\[ 1 \leq M \leq N^3, \quad MN \leq q. \]
Then
\[ B(K, \tilde{\alpha}, 1_N) \ll q^{\epsilon} ||\tilde{\alpha}||_1^{2/3} ||\tilde{\alpha}||_2^{1/3} M^{1/6} N \left( \frac{q^4}{M^{1/3} N^2} \right)^{1/24}. \]

This defeats the trivial estimate
\[ B(K, \tilde{\alpha}, 1_N) \ll N ||\tilde{\alpha}||_1 \ll ||\tilde{\alpha}||_1^{2/3} ||\tilde{\alpha}||_2^{1/3} M^{1/6} N, \]
as soon as \( M = N \geq q^{2/5 + \delta} \), say.

2. Proof of Theorem 1.4

To prove Theorem 1.4, we begin as in [?, §2] with the ‘+ab-shifting’ trick. Given parameters \( A, B \geq 1 \) such that
\[ AB \leq N, \quad AM < q, \]
we have
\[ B(K, \tilde{\alpha}, N) \ll \frac{q^\epsilon}{AB} \sum_{r \bmod q} \sum_{s \leq 2AM} \nu(r, s) \mu(r, s) \]
where
\[ \nu(r, s) = \ldots \]
(note the \( N \) here should be \( N' \), an interval of length \( \leq 2N \)) and
\[ \mu(r, s) = \left| \sum_{B < b \leq 2B} \eta_B K(s(r + b)) \right|. \]
For \( \nu \), we have the moment estimates
\[
||\nu||_1 \ll AN||\tilde{\alpha}||_1
\]
and
\[
||\nu||_2^2 \ll q^\epsilon AN||\tilde{\alpha}||_2^2
\]
from [?].

Now we apply Hölder’s inequality with exponent 6:
\[
\sum_{r \mod q \atop s \leq 2AM} \nu(r,s)\mu(r,s) = ||\nu\mu||_1 \\
\leq ||\nu^{2/3}||_{3/2}||\nu^{1/3}||_6||\mu||_6 \\
\ll (AN||\tilde{\alpha}||_1)^{2/3}(q^\epsilon AN||\tilde{\alpha}||_2^{1/3})^{1/6}||\mu||_6
\]

We adapt the standard notational convention that \( \epsilon \) denotes an arbitrarily small positive number, whose value may differ between instances. After a small amount of bookkeeping, we now have

\[
B(K,\tilde{\alpha},N) \ll \frac{q^\epsilon}{AB} (AN)^{\frac{5}{6}}||\tilde{\alpha}||_1^{2/3}||\tilde{\alpha}||_2^{1/3}||\mu||_6.
\]

By the triangle inequality, we have
\[
||\mu||_6^6 \leq \sum_{\tilde{b} \in \mathcal{B}} |S(K,\tilde{b}; 2AM)|
\]
where
\[
\mathcal{B} = \{\tilde{b} \in \mathbb{Z}^6 : B < b_i \leq 2B, 1 \leq i \leq 6\}
\]
and
\[
S(K,\tilde{b}; 2AM) = \sum_{r \mod q \atop s_1 = 1} \prod_{i=1}^{3} K(s(r + b_i))\overline{K}(s(r + b_{i+3})).
\]

Here \( K(x) = \overline{K}(x) \).

**Definition 2.1.** Let \( \mathcal{V}^\Delta \) be the affine variety of sextuples
\[
\tilde{b} = (b_1, \ldots, b_6) \in \mathbb{A}_F^6
\]
defined by the conditions
\[
\begin{align*}
(1) & \text{ If } k \text{ is even, then for any } i \in \{1, \ldots, 6\} \text{ the cardinality } \#\{j : b_j = b_i\} \text{ is even.} \\
(2) & \text{ If } k \text{ is odd and not equal to } 3, \text{ then } \{\{b_1,b_2,b_3\}\} = \{\{b_4,b_5,b_6\}\} \text{ is an equality of multisets.} \\
(3) & \text{ If } k = 3, \text{ then either } \{\{b_1,b_2,b_3\}\} = \{\{b_4,b_5,b_6\}\} \text{ or } \tilde{b} = (b,b,b,b',b',b') \\
& \text{ for some } b,b'.
\end{align*}
\]

The role of the ‘diagonal set’ is played by
\[
\mathcal{B}^\Delta = \{\tilde{b} \in \mathcal{B} : \tilde{b} \mod q \in \mathcal{V}^\Delta\}.
\]

Observe that the contribution from the vectors \( \tilde{b} \in \mathcal{B}^\Delta \) to \( ||\mu||_6^6 \) satisfies
\[
\sum_{\tilde{b} \in \mathcal{B}^\Delta} |S(K,\tilde{b}; 2AM)| \ll qAB^3M := x_1.
\]
For \( \tilde{b} \not\in B^\Delta \), we can exploit averaging over \( r \):

**Lemma 2.2.** For \( b \in B \setminus B^\Delta \) and \( s \in F_q^\times \), we have

\[
\sum_{r \mod q} \prod_{i=1}^{3} K(s(r+b_i)) \overline{K}(s(r+b_{i+3})) \ll q^{1/2}.
\]

In particular, for any \( B' \subset B \setminus B^\Delta \) we have

\[
\sum_{\tilde{b} \in B'} |S(K, \tilde{b}, 2AM)| \ll AMq^{1/2}|B'|.
\]

We refer to §3 for the proof. Generically we’ll need to save more than \( q^{1/2} \).

An application of the Plancherel formula—this is the Pólya-Vinogradov method from §4 of our course notes—yields

\[
S(K, \tilde{b}, 2AM) \ll (\log q) \max_{\lambda \mod q} |\hat{S}(K, \tilde{b}, \lambda)|
\]

where

\[
\hat{S}(K, \tilde{b}, \lambda) = \sum_{r \mod q} R(K, r, \lambda, \tilde{b})
\]

with

\[
R(K, r, \lambda, \tilde{b}) = R(K, r, \lambda, \hat{b}) = \sum_{s \mod q} e_q(\lambda s) \prod_{i=1}^{3} K(s(r+b_i)) \overline{K}(s(r+b_{i+3})).
\]

By following the proof of [? , Theorem 2.3], we obtain the following generic estimate.

**Theorem 2.3.** There exists a codimension one subvariety \( V^{\text{bad}} \subset A_{F_q}^6 \) containing \( V^\Delta \), with degree bounded independently of \( q \), such that if \( \lambda \in F_q \) and \( \tilde{b} \not\in V^{\text{bad}}(F_q) \) then \( \hat{S}(K, \tilde{b}, \lambda) \ll q \) and therefore \( S(K, \tilde{b}, 2AM) \ll q \log q \).

This uses the full power of Deligne-Katz [?], but an improvement could still be sought on the codimension.

Write

\[
B^{\text{bad}} = \{ \tilde{b} \in B : \tilde{b} \mod q \in V^{\text{bad}}(F_q) \}
\]

and

\[
B^{\text{gen}} = B \setminus B^{\text{bad}}.
\]

By Schwartz-Zippel and uniformity of the degree bound, we have \#\( B^{\text{bad}} \leq (\deg V^{\text{bad}}) B^5 \ll B^5 \). Thus by Lemma 2.2 we have

\[
\sum_{\tilde{b} \in B^{\text{bad}} \setminus B^\Delta} |S(K, \tilde{b}; 2AM)| \ll q^{1/2}AB^5M := x_2.
\]

By Theorem 2.3 we have

\[
\sum_{\tilde{b} \in B^{\text{gen}}} |S(K, \tilde{b}; 2AM)| \ll (\log q)qB^6 := (\log q)x_3.
\]

Thus

\[
\|\mu\|_6^6 \ll (x_1 + x_2 + x_3) \log q
\]

where \( x_1 = qAB^3M, x_2 = q^{1/2}AB^5M, x_3 = qB^6 \).
Choosing
\[ A = M^{-1/4}N^{3/4}, \quad B = M^{1/4}N^{1/4} \]
ensures that \( AB = N \) and \( x_1 = x_3 \).

We note that the hypotheses of our theorem ensure that
\[ A \geq 1, \quad AM < q \]
as our parameters are acceptable. Moreover, the hypothesis \( MN \leq q \) ensures that \( x_2 \leq x_3 = q(MN)^{3/2} \).

Now from (1) we have
\[
B(K, \tilde{\alpha}, \mathcal{N}) \ll \frac{q^\epsilon}{N}(AN)^{5/6}||\tilde{\alpha}||_{2}^{2/3}||\alpha||_{1}^{1/3}q^{-1/6}(MN)^{1/4}
\]
\[
= \frac{q^{\epsilon+1/6}}{N}N^{-5/6}\left(M\right)^{1/24}||\tilde{\alpha}||_{1}^{2/3}||\alpha||_{2}^{1/3}(MN)^{1/4}
\]
\[
= q^{\epsilon+1/6}M^{1/24}N^{17/24}||\tilde{\alpha}||_{1}^{2/3}||\alpha||_{2}^{1/3}
\]
\[
= q^{\epsilon}M^{1/6}N||\tilde{\alpha}||_{1}^{2/3}||\alpha||_{2}^{1/3}\left(\frac{q^{4}}{M^{3}N^{3}}\right)^{1/24}
\]

We use a similar strategy to prove Theorem 1.2. This time Cauchy-Schwartz, the +ab-shifting trick, and Hölder-6 give
\[
B(K, \tilde{\alpha}, \tilde{\beta})^2 \ll ||\tilde{\alpha}||_{2}^{2}||\tilde{\beta}||_{2}^{2}(N + \frac{q^\epsilon}{AB}M^{2/3}(AN)^{5/6}||\mu'||_{6}),
\]
where
\[
||\mu'||_{6} = \sum_{\theta \in \mathcal{B}}|S^{\theta}(K, \tilde{b}; 2AM)|.
\]

Here
\[
S^{\theta}(K, \tilde{b}; 2AM) = \sum_{r \mod q \atop s_1 \neq s_2 \mod q \atop 1 \leq i \leq 3} K(s_1(r+b_i))K(s_2(r+b_i))K(s_1(r+b_{i+3}))K(s_2(r+b_{i+3})).
\]

In §3, we shall also confirm the following analogue of Lemma 2.2:

**Lemma 2.4.** For any subset \( \mathcal{B}' \subset \mathcal{B} \setminus \mathcal{B}^\Delta \) we have
\[
\sum_{\theta \in \mathcal{B}'} |S^{\theta}(K, \tilde{b}; 2AM)| \ll (AM)^{2}q^{1/2}|\mathcal{B}'|.
\]

For \( \mathcal{B}^\Delta \), we have the trivial bound
\[
(2) \quad \sum_{\theta \in \mathcal{B}'} |S^{\theta}(K, \tilde{b}; 2AM)| \ll qA^{2}B^{3}M^{2} := y_1.
\]

We WHAT the condition \( s_1 \neq s_2 \mod q \) by the indicator function expression,
\[
1 - \frac{1}{q} \sum_{\lambda \mod q} e\left(\lambda(s_1 - s_2)\right).
\]

The Pólya-Vinogradov method then gives
\[
S^{\theta}(K, \tilde{b}; 2AM) \ll (\log q)^{2} + (\log q)^{2} \max_{\lambda_1, \lambda_2 \mod q} |\tilde{S}(K, \tilde{b}, \lambda_1, \lambda_2)|.
\]
Theorem 2.5. There exists a codimension one subvariety \( V^\Delta \subset \mathbb{A}^6_F \) containing \( V \), with degree bounded independently of \( q \), such that for every \( \tilde{b} \notin V^\Delta(F_q) \) and every distinct \( \lambda_1, \lambda_2 \in F_q \), we have
\[
\tilde{S}(K, \tilde{b}, \lambda_1, \lambda_2) = \zeta(\lambda_1, \lambda_2, \tilde{b}) - \frac{1}{q} \sum_{\lambda \mod q} \zeta(\lambda_1 + \lambda, \lambda_2 + \lambda, \tilde{b}),
\]
where
\[
\zeta(\lambda_1, \lambda_2, \tilde{b}) = \sum_{r \mod q} R(r, \lambda_1, \tilde{b})R(r, \lambda_2, \tilde{b}).
\]

Mimicking the proof of [KMS, Theorem 2.5] gives

**Theorem 2.5.** There exists a codimension one subvariety \( V^\Delta \subset \mathbb{A}^6_F \) containing \( V \), with degree bounded independently of \( q \), such that for every \( \tilde{b} \notin V^\Delta(F_q) \) and every distinct \( \lambda_1, \lambda_2 \in F_q \), we have
\[
\tilde{S}(K, \tilde{b}, \lambda_1, \lambda_2) \ll q^{3/2}.
\]

In fact, \( V^\Delta \) is the same in Theorem 2.3 and Theorem 2.5. Using Lemma 2.4 for \( \tilde{b} \notin B^{\text{gen}} \) gives
\[
||\mu||_B^2 \ll (\log q)^2(y_1 + y_2 + y_3)
\]
where
\[
y_1 = qA^2B^3M^2, \quad y_2 = q^{1/2}A^2B^5M^2, \quad y_3 = q^{3/2}B^6.
\]
Choosing
\[
A = q^{1/3}M^{-2/3}N^{1/3}, \quad B = q^{-1/3}M^{2/3}N^{2/3},
\]
we have
\[
AB = N, \quad y_2 = y_3.
\]

Moreover, the hypothesis \( MN \geq q^{7/8} \) implies that \( y_1 \leq y_3 = q^{-1/2}M^4N^4 \). Now
\[
\frac{B(K, \tilde{\alpha}, \tilde{\beta})}{||\tilde{\alpha}||_2||\tilde{\beta}||_2} \ll \sqrt{N} + q^e\left(\frac{M^{2/3}(AN)^{5/6}}{AB}\right)^{1/2}\left(\frac{M^4N^4}{q^{1/2}}\right)^{1/12}
\]
\[
= \sqrt{N} + q^{-1/24}M^{1/3}(qM^{-2}N^4)^{5/36}M^{1/3}N^{1/6}
\]
\[
= \sqrt{N} + q^{e+7/72}M^{7/18}N^{7/18}
\]
\[
= \sqrt{N} + q^e(MN)^{1/2}(q^7(MN)^{-8})^{1/72}.
\]

Thus
\[
B(K, \tilde{\alpha}, \tilde{\beta}) \ll q^e||\tilde{\alpha}||_2||\tilde{\beta}||_2(MN)^{1/2}(M^{-1/2} + q^7(MN)^{-8})^{1/72}.
\]

3. PROOF OF LEMMAS

**Proof of Lemma 2.2.** We appeal directly to [?, Corollary 1.6]. The relevant vector is
\[
\tilde{\gamma} = (\gamma_{s,1}, \ldots, \gamma_{s,6}),
\]
where
\[
\gamma_{s,i} = \begin{pmatrix} s & sb_i \\ 0 & 1 \end{pmatrix},
\]
for \( i = 1, \ldots, 6 \).

- **Case 1:** \( k \) even. If \( \tilde{b} \notin B^\Delta \) then there exists an \( i \) such that \( \#\{j : b_j = b_i\} \) is odd.
Case 2: \( k > 3 \) odd. Here \( \tilde{\sigma} = (1, 1, 1, c, c, c) \), where \( c \) denotes complex conjugation. If \( \tilde{b} \not\in B^A \), then
\[
\{\{b_1, b_2, b_2\}\} \neq \{\{b_4, b_5, b_6\}\}.
\]
In particular, there exists an \( i \) such that
\[
\#\{j : b_i = b_j, j \geq 3\} \neq \#\{j : b_i = b_j, j \geq 4\}.
\]
As known from [5, Remark 1.9], the special involution is
\[
\xi = \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}.
\]
Since \( q \neq 2 \) and \( s \in F_q^\times \), we can never have \( \xi \gamma_i = \gamma_j \). The conditions (2) and (3) of [5, Definition 1.3] are thus equivalent.

Case 3: \( k = 3 \) This is almost the same as Case 2. The only thing that could go wrong is if \( \beta = (b, b, b, b', b', b') \), but we have explicitly excluded this.

\[\square\]

**Proof of Lemma 2.4.** We have
\[
\tilde{\gamma} = (\gamma_{s_1,1}, \ldots, \gamma_{s_1,6}, \gamma_{s_2,1}, \ldots, \gamma_{s_2,6})
\]
and
\[
\tilde{\sigma} = (1, 1, 1, c, c, c, c, c, 1, 1, 1).
\]
Recall that \( s_1 \neq s_2 \) mod \( q \).

- Case 1: \( k \) even. If \( \tilde{b} \not\in B^A \) then there exists an \( i \) such that \( \#\{j : b_j = b_i\} \) is odd. Thus \( \#\{j \leq 12 : \gamma_j = \gamma_i\} \) is odd; here \( \gamma_i = \gamma_{s_1,i} \), since \( i < b \).
- Case 2: \( k > 3 \) odd. If \( \tilde{b} \not\in B^A \) then
\[
\#\{j : b_i = b_j, j \geq 3\} \neq \#\{j : b_i = b_j, j \geq 4\},
\]
so there exists \( i \leq 6 \) such that
\[
\#\{j \leq 3 : b_j = b_i\} \neq \#\{j \geq 4 : b_j = b_i\}.
\]
and note that \( \gamma_j \neq \gamma_i \) for \( j > 6 \) (as \( s_1 \neq s_2 \) mod \( q \)). Since \( k > 3 \), this also means that the two expressions are incongruent modulo \( k \).

Also, \( k \)-normality of \((\tilde{\gamma}, \tilde{\sigma})\) is the same with or without respect to the special involution \( \xi \), since we can never have \( \xi \gamma_i = \gamma_j \). To see this, note that if \( \xi \gamma_i = \gamma_j \) then
\[
q|2s_i \text{ or } q|(s_1 + s_2),
\]
for some \( i = 1, 2 \), both of which are impossible since \( q \neq 2, s_i \in F_q^\times \), and
\[
4AM = 4(qMN)^{1/3} \leq q \iff 64MN \leq q^2
\]
the latter given by our hypothesis.

- Case 3: \( k = 3 \). Again this is basically the same as Case 2, since we’ve explicitly forbidden vectors \( b \) of the shape \((b, b, b, b'b'b')\).

\[\square\]