# HOMOTOPY METHODS IN AUTOMORPHIC FORMS [DRAFT]

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Contents



## 1. Introduction

Beginning with the Adams conjecture in the homotopy groups of spheres, there has been a curious overlap between homotopy theory and number theory. For example, special values of $\zeta(s)$ appear in the study of homotopy groups of spheres, while motivic $L$-functions, defined using motivic cohomology or algebraic $K$-theory, give the most general statement of the BSD conjecture. With the introduction of commutative spectra, topological automorphic forms, and motivic homotopy theory, the picture gets curiouser still. But as of yet it is unclear what the connection is precisely, and perhaps more philosophically, why should they be connected?

On the other hand, recent advances in the Langlands program suggest the utility of methods that go beyond the usual methods of representation theory and algebraic geometry, including that of perfectoid spaces, covering groups, and higher categories. Indeed, the methods of homotopy theory have made a powerful impact in algebraic geometry, thus it is natural to ask what homotopical methods can do for representation theory, and in particular automorphic forms.

The goal of this paper is twofold: First, to survey the instances where methods beyond algebraic geometry have appeared in arithmetic, and where number theory has appeared in homotopy theory. Second, based on these clues, to offer ideas as to what the Langlands conjectures in the arithmetic setting might look like in the topological picture, and what homotopical methods might do for number theory. It is our hope that this will make the situations where arithmetic appears—sometimes unexpectedly—in algebraic topology more well known amongst number theorists, and to suggest as does Bernstein implicitly in [2], the usefulness of homotopy theory in the theory of automorphic forms.

The paper is organized as follows: Section 2 reviews the Langlands reciprocity conjecture, also known as the Langlands correspondence, and surveys recent advances where methods beyond algebraic geometry have entered into the picture. In particular, we focus on

(1) Scholze's work on torsion in the cohomology of locally symmetric varieties,
(2) Weissman's framework for a Langlands correspondence for covering groups,
(3) Kapranov's Langlands correspondence for $n$-dimensional local fields.

Section 3 explores the overlap between homotopy theory and arithmetic. In particular, we discuss

(1) The Adams conjecture and conjectures of Beilinson, Lichtenbaum, and others on special values of $L$-functions,
(2) Morel and Voevodsky's motivic (stable) homotopy theory of schemes,
(3) Behrens and Lawson's spectrum of topological automorphic forms, and chromatic homotopy theory in general.

Finally, in Section 4 we suggest constructions applying homotopy theory to the Langlands reciprocity conjecture, and speculate regarding the arithmetic nature of chromatic homotopy theory. In particular, we replace the category of representations of $G$ with the model structure of simplicial presheaves on the stack $BG$, and study the underlying $(\infty, 1)$-category, and show that it is compatible with Kapranov's $n$-dimensional Langlands correspondence.

It is worth noting that there are other connections which we do not consider in this paper, for example: the Connes-Consani approach to the Riemann zeta function through cyclic homology, complemented by recent work of Hesselholt, the Fontaine-Fargue curve in $p$-adic Hodge theory, and also the recent construction of integral $p$-adic Hodge theory by Bhatt-Morrow-Scholze.

In order to make the paper accessible to both algebraic topologists and number theorists, we will belabor the definitions on both sides, assuming little background in each.

## 2. Langlands reciprocity beyond algebraic varieties

2.1. **Reciprocity from Legendre to Langlands.** Let's begin with the classical quadratic reciprocity law, coined by Legendre and proved (many times) by Gauss. Recall that the Legendre symbol $(\frac{d}{p})$ for any odd prime $p$ and any integer $d$ prime to $p$ is defined to be 1 if $d$ is a square mod $p$ and $-1$ otherwise. Then the quadratic reciprocity law can be stated as follows:

**Theorem 2.1.1** (Quadratic reciprocity law)**.** *Let $p, q$ be distinct odd primes. Then*

$$\left(\frac{p}{q}\right) = (-1)^{(p-1)/2\cdot(q-1)/2}\left(\frac{q}{p}\right),$$

*Put differently, the splitting behaviour of a monic irreducible $x^2 - q$ mod $p$ is determined by the congruence class of $p$ mod $4q$.*

Subsequent work of Eisenstein, Kummer, Takagi, Hilbert, Artin, and others proved higher reciprocity laws, culminating in what is referred to as abelian class field theory, which studies the splitting behaviour of primes in abelian extensions of number fields.

The main theorem of class field theory is Artin's reciprocity law over global fields and local fields. In order to state the theorem we have to introduce the norm residue symbol generalizing the Legendre symbol, which we do not define here, and the ideal class group $C_F$ of fractional ideals modulo principal ideals of $F$.

**Theorem 2.1.2** (Artin's reciprocity law)**.** *(Global) Let $K/F$ be a finite extension of number fields. Then the global norm residue symbol induces an isomorphism*

$$C_F/N_{K/F}C_K \simeq \mathrm{Gal}(K/F)^{ab}.$$

*(Local) Let $K/F$ a finite extension of finite $\mathbf{Q}_p$extensions. Then the local norm residue symbol induces an isomorphism*

$$F^\times/N_{K/F}K^\times \simeq \mathrm{Gal}(K/F)^{ab}.$$

*Put differently, there exists an ideal $\mathfrak{q}$ of $\mathcal{O}_k$, such that if a principal prime ideal $\mathfrak{p} = (\pi)$ of $k$ satisfies $\pi \equiv 1 \mod \mathfrak{q}$ and $\sigma(\pi) > 0$ for all real embeddings $\sigma$ of $k$, then $\mathfrak{p}$ splits in $K$.*

Following this, the search continued for a nonabelian recicprocity law, a rule for determining the splitting behaviour of primes in nonabelian extensions of number fields. The most successful generalization, arguably, is Langlands reciprocity, often referred to as the Langlands correspondence. Taking the inverse limit of finite extensions $K$ of $F$, Artin reciprocity law leads to the isomorphism

$$F^\times\backslash\mathbf{A}_F^\times \simeq \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$$

where $\mathbf{A}_F = \prod_v{}' F_v$ denotes the adéle ring of $F$, which is the restricted direct product over completions $v$ of $F$, 'restricted' meaning that at almost every $v$ we take the ring of integers $\mathcal{O}_{F_v}$.

Langlands interprets the bijection in terms of characters, i.e., one-dimensional representations:

$$\{GL_1(F)\backslash GL_1(\mathbf{A}_F) \to \mathbf{C}\} \longleftrightarrow \{\mathrm{Gal}(\bar{F}/F) \to GL_1(\mathbf{C})\}$$

More generally, the group $GL_1$ is replaced with an arbitrary reductive linear algebraic group $G$ on the left, and imposing certain conditions one obtains automorphic forms $f$ on $G$, while on the right one has the associated $L$-group of ${}^LG$. Note that instead of $\mathbf{C}$ we could have used any algebraic closure of $\mathbf{Q}$. Langlands' reciprocity conjecture is can be stated roughly as follows:

**Conjecture 2.1.3** (Langlands reciprocity)**.** *Let $f$ be a cuspidal algebraic automorphic form on $GL_n$. Fix a prime $p$ and an isomorphism $\mathbf{C} \simeq \mathbf{Q}_p$. Then there exists an irreducible $p$-adic Galois representation*

$$\rho : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_n(\bar{\mathbf{Q}}_p)$$

*such $\rho$ is unramified at almost every prime $l$. Moreover, $\rho$ is geometric, i.e, it occurs as a subquotient of $H^i_{\acute{e}t}(X \times_{\mathbf{Q}} \bar{\mathbf{Q}}, \mathbf{Q}_p)$ for some projective, nonsingular algebraic variety $X$ over $\mathbf{Q}$.*

Here we have stated the conjecture over $\mathbf{Q}$, for simplicity, rather than for a general number field. There is a converse to this statement, known as the Fontaine-Mazur conjecture. The condition that $\rho$ is geometric can be interpreted as the Galois representation arising from a *motive*, which we shall discuss shortly.

A major advance in this direction is what is referred to the Modularity Theorem. To describe the theorem we introduce the $p$-adic Tate module of an elliptic curve $E$,

$$T_pE = \varprojlim_m E[p^m] \simeq \mathbf{Z}_p^2$$

which defines a $p$-adic Galois representation $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathbf{Q}_p)$. Note that in contrast to Artin representations, which are continuous representations $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to$

$GL_n(\mathbf{C})$, $p$-adic Galois representations have infinite image. Also define the modular curve $X_1(N) = \Gamma_1(N)\backslash\mathbf{H}$, where

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \;\middle|\; a, d \equiv 1 \mod N, c \equiv 0 \mod N \right\}$$

is the usual congruence subgroup. Note that $\mathbf{H} \simeq SL_2(\mathbf{R})/SO_2$. The modular curve is a moduli space of elliptic curves over $\mathbf{Q}$ with $N$ level structure, On the other hand, automorphic forms on $GL_2$ over $\mathbf{Q}$ can be realized as modular forms on $X_1(N)$. We call a modular form $f$ cuspidal if it vanishes on the cusps of $X_1(N)$, viewed as a Riemann surface, and an eigenform if it is an eigenvector with respect to the action of the Hecke operators $T_p$ acting on modular forms for every $p$.

We now state the reciprocity law for elliptic curves over $\mathbf{Q}$ as follows:

**Theorem 2.1.4** (Shimura-Taniyama-Weil conjecture)**.** *Let $E$ be an elliptic curve over $\mathbf{Q}$ with level structure $N$, and $p$ a prime. Then there exists a weight $k$ cuspidal eigenform $f$ and character $\chi$ such that for almost all primes $l$, the characteristic polynomial of $Frob_l$ acting on $T_pE$ is equal to*

$$x^2 - a_l(f)x + \chi(l)l^{k-1}$$

*where $a_l(f)$ is the Hecke eigenvalue of $T_l$ acting on $f$.*

*In other words, every elliptic curve over $\mathbf{Q}$ is modular.*

This was proved in special cases by Wiles and Taylor, and completed by [**?**, BCDT] The algebraic and analytic sides are connected through the comparison isomorphism

$$H^1(X_1(N), \mathbf{Q}) \otimes \mathbf{Q}_p \simeq H^1_{\text{ét}}(X_1(N), \mathbf{Q}_p)$$

of singular and étale cohomology, and crucial to the proof was the fact that the compactification of $X_1(N)$ exists as an algebraic variety defined over $\mathbf{Q}$. (Indeed, it is a complex Riemannian manifold with finitely many punctures.)

2.2. **The cohomology of locally symmetric varieties.** To generalize the picture above, we would like to consider, for example, the quotient $X = \Gamma\backslash SL_n(\mathbf{R})/SO_n$ where $\Gamma$ is a congruence subgroup of $SL_n(\mathbf{Z})$ defined analogous to $\Gamma_1(N)$. The action of Hecke operators are still defined on $H^i(X, \mathbf{Q})$, however, the space $X$ is no longer a complex manifold for $n > 2$. Thus the methods of finding Galois representaitons as in the previous case do not generalize immediately. Nonetheless, the following theorem was recently proved:

**Theorem 2.2.1** (HLTT, Scholze)**.** *Let $f$ be a Hecke eigenclass in $H^i(X, \mathbf{C})$, and let $a_l(f)$ be the eigenvalue of the Hecke operator $T_{l,k}$ on $f$ for $l \not| N$, and $k = 1, \dots n$. Then there exists a continuous semisimple $p$-adic Galois representation*

$$\rho : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_n(\bar{\mathbf{Q}}_p)$$

*such that $\rho$ is unramified at all $l$ and the characteristic polynomial of $\rho(\mathrm{Frob}_l)$ is*

$$x^n + \sum_{k=1}^{n} (-1)^k l^{i(k-1)/2} a_{l,k}(f) x^{n-k}. \tag{2.1}$$

*Moreover, such a correspondence holds for cuspidal algebraic automorphic forms on $GL_n$ over $F$ totally real or CM.*

To discuss this result, we introduce the notion of a Shimura variety: Let $G$ be a semisimple algebraic group over $\mathbf{Q}$, $K$ a maximal compact subgroup of the Lie group $G(\mathbf{R})$, and $\Gamma$ an arithmetic subgroup of $G_(\mathbf{R})$ generalizing finite-index subgroups of $SL_2(\mathbf{Z})$. The quotient $\Gamma\backslash G(\mathbf{R})/K$ is a locally symmetric Riemannian space, which we call an arithmetic manifold. In particular, we consider $\Gamma$ which contain $\ker(G(\mathbf{Z}) \to G(\mathbf{Z}/N\mathbf{Z}))$ for some $N$, in which case we call $\Gamma$ a congruence subgroup. If the quotient $G(\mathbf{R})/K$ has a complex structure compatible with the Riemannian metric, it is a Hermitian symmetric domain, and in many cases the resulting quotient $X$ is an algebraic variety defined over a number field, which we call a Shimura variety.

An important class of examples are the Siegel modular varieties,

$$Sh = \Gamma\backslash Sp_{2n}(\mathbf{R})/U(n)$$

where $Sp_{2n}$ is the group of symplectic $n \times n$ matrices. In this case there is a correspondence between Siegel cuspidal eigenforms with $GL_{2n+1}(\bar{\mathbf{Q}}_p)$ Galois representations.

Suppose $X$ is an arithmetic manifold for $GL_n$, which we remind the reader is not an algebraic variety in general. The Borel-Serre compactification $Sh^{\mathrm{BS}}$ of $Sh$ enjoys the following properties:

- The compactification $Sh^{\mathrm{BS}}$ is a manifold with corners,
- The inclusion $Sh \hookrightarrow Sh^{\mathrm{BS}}$ is a homotopy equivalence,
- The boundary $Sh^{\mathrm{BS}}\backslash Sh$ is a stratified manifold, with each stratum a locally symmetric space for a parabolic subgroup of $Sp_{2n}$,
- The boundary contains a torus bundle $X_P$ over $X$ as an open subset

As a result, cohomology classes on $X$ appear in $Sh$, and the compactification preserves Hecke eigenclasses, so that given a Hecke eigenclasses in $H^i(X, \mathbf{F}_p)$ there exists one in $H^i(Sh, \mathbf{F}_p)$ whose Hecke eigenvalues are related to the first in a systematic way.

Automorphic forms that can be detected through the action of Hecke operators on cohomology, using Matsushima's formula, are called cohomological. If the eigenclass $g$ in $H^i(Sh, \bar{\mathbf{F}}_p)$ were the image of a torsion-free eigenclass in $H^i(Sh, \mathbf{Z})$, then we would be able to find the corresponding automorphic form like so. However, this may not be the case, but Scholze's theorem shows that the correspondence holds mod $p$:

**Theorem 2.2.2** (Scholze). *Let $g$ be a Hecke eigenclass in $H^i(X, \mathbf{F}_p)$, then there exists a Siegel cusp form $h$, possibly for a subgroup of $\Gamma$, whose Hecke eigenvalues are congruent to those of $g$ mod $p$.*

*As a result, there exists a continuous semisimple $p$-adic Galois representation*

$$\rho : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_n(\bar{\mathbf{F}}_p)$$

*as above, with the coefficients of (2.1) in $\mathbf{F}_p$.*

Regarding Scholze's proof, we only mention that his method uses a pro-étale cover of $Sh$ as a rigid-analytic space. It is a Shimura variety 'at infinite level' $Sh^{\mathrm{an}}_{p^\infty}$, which exists as a perfectoid space, and admits a period morphism to the Grassmanian $Gr(n, 2n)$, also viewed as a perfectoid space.

2.3. **The case of covering groups.** The conjectures of Langlands were made for automorphic forms on reductive linear algebraic groups, as the natural generalization of the representation theory of compact Lie groups, most of all $SL_2(\mathbf{R})$. As algebraic groups they satisfy good functorial properties like base change and restriction, and being reductive they behave well with respect to induction from Levi components of parabolic subgroups, being themselves reductive.

On the other hand, nonlinear covering groups, which are often no longer algebraic, have appeared in classical considerations. Half integral weight modular forms..

**Theorem 2.3.1** (Shimura lift)**.**

The metaplectic group $Mp_{2n}$, which is the double cover of $Sp_{2n}$ has a family of distinguished representations known as the Weil representation.

**Theorem 2.3.2** (Weil representation)**.**

For every simply-connected isotropic simple algebraic group $G$ over a field $F$, there is a universal abstract extension

$$1 \to K_2(F) \to E \to G(F) \to 1$$

where $K_2(F)$ is the Milnor $K$-group of $F$, isomorphic to $F^\times \otimes F^\times$ modulo the relations $(x, 1-x)$ and $(x, x)$. It is universal in the sense that given any abelian group $A$, one has $H^2(G, A) = \mathrm{Hom}(K_2(F), A)$. Unfortunately, once we loosen the assumptions on $G$, there is no longer a universal cover, and the extension does not behave well functorially.

To address this, Weissman suggests the extension by multiplicative $K_2$-torsors on $G$ defined by Brylinski and Deligne, now referred to as BD-extensions:

$$1 \to K_2 \to E \to G \to 1$$

which, after taking $F$-points and pushing out by the Hilbert symbol (as in the Hilbert reciprocity law), one has

$$1 \to \mu_n(F) \to \tilde{G} \to G(F) \to 1$$

where $\mu_n(F)$ denotes the $n$-th roots of unity in $F$. This singles out a class of topological covering groups which appear to be better behaved functorially, and includes covers of all simply-connected $G$. For this class of covering groups, [**?**] and [**?**] have defined $L$-groups by which we may define a Langlands reciprocity.

We close this section with a recent result: For a BD-extension $\widetilde{G}$, fix an embedding of $\mu_n$ into $\mathbf{C}^\times$, and call a representation $\pi$ of $G$ a genuine representation if $\pi|_{\mu_n} = \epsilon$. Then one has a local result

**Theorem 2.3.3.** [**?**] *Let $\psi$ be a nontrivial additive character of $F$, then there is a bijection depending on $\psi$,*

$$\mathrm{Irr}_\epsilon(Mp_{2n}) \leftrightarrow \mathrm{Irr}(SO_{2n+1}(F) \sqcup \mathrm{Irr}(SO_{2n+1}(F).$$

And the corresponding global result

**Theorem 2.3.4.** [**?**] *The elliptic endoscopic groups of $Mp_{2n}$ are $SO_{2a+1} \times SO_{2b+1}$ where $a+b=n$, and there exists transfer factors*

$$\Delta_{a,b} : Mp_{2n} \times (SO_{2a+1} \times SO_{2b+1}) \to \mathbf{C}$$

*and a transfer of test functions inducing a transfer of distriibutions*

$$D(SO_{2a+1} \times SO_{2b+1}) \to D(Mp_{2n})$$

*by matching orbital integrals.*

Efforts to incorporate the theory of covering groups is still in its beginnings, and there are many questions yet to be answered,

2.4. **Reciprocity for higher dimensional local fields.** We set the scene with what is referred to as the local Langlands correspondence, which we have called Langlands reciprocity over local fields.

**Theorem 2.4.1** ([**?**, **?**])**.**

Now recall that a nonarchimdean local field is a complete discrete valued field with finite residue field, e.g., a finite extension of $\mathbf{Q}_p$ or $\mathbf{F}_p((t))$. Then we inductively define an *n-dimensional local field* to be one whose residue field is an $(n-1)$ dimensional local field, e.g., $\mathbf{Q}_p((t))$ and $\mathbf{F}_p((t_1))((t_2))$. From work of Parshin, we note that these arise naturally as completions of fraction fields on schemes of absolute dimension $n$.

## 3. Arithmetic in the homotopy theory of schemes and spheres

We will discuss the following clues from homotopy theory

(1) *The Adams conjecture.* The image of the classical $J$-homomorphism

$$\pi_n(SO_k) \to \pi_{n+k}(S^k)$$

is generated by homotopy elements $\alpha_{i/j}$ in $\pi_* S$, which are detected in the 0-line of the Adams-Novikov spectral sequence for the $n$-th monochromatic layer $M_n S$. These elements are related to the $p$-adic valuations of denominators of Bernoulli numbers.

On the other hand, congruences of Bernoulli numbers, viewed as special values of the Riemann zeta function allow us to $p$-adically interpolate $\zeta(s)$ to a $p$-adic analytic function.

(2) *The homotopy theory of schemes.* The work of Morel and Voevodsky develops the stable motivic homotopy theory of schemes, producing a model category from which Voevodsky is able to solve the Milnor and Bloch-Kato conjectures, the latter requiring the contribution of many others. Robalo follows this up by describing the $(\infty, 1)$-category underlying the model structure. In light of the Langlands reciprocity conjecture, one expects a functor from the category of motives to the category automorphic forms, at least the $L$-algebraic (in the sense of Clozel) isobaric automorphic forms on $GL_n$.

How should the category of automorphic forms be realized as a stable homotopy category? Or might it be necessary to delve deeper into derived schemes to find a proper analog on the automorphic side.

(3) *Topological automorphic forms and chromatic homotopy theory.* On the other hand, the Morava stabilizer group $S_n$ is the automorphism group of the height $n$ Honda formal group law, which is a Lubin-Tate formal group, over $\mathbf{F}_p$, whence the extended Morava stabilizer group $G_n = S_n \rtimes \mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p)$. One observes that $S_n \simeq \mathfrak{o}_D^\times$, the ring of integers of a $\mathbf{Q}_p$-central division algebra. Division algebras are inner forms of $GL_n$, and by the Jacquet-Langlands correspondence there is a functorial transfer of automorphic forms from one to the other. By Morava's change of rings theorems the Adams-Novikov spectra sequence for the $K(n)$ local sphere gives

$$H_c^*(G_n, (E_n)_*) \Rightarrow \pi_* S_{K(n)}$$

The memoir by Behrens and Lawson on TAF spectra on Shimura stacks associated to $U(n-1, 1)$ generalize the TMF spectra developed by Hopkins and Miller. The key point here is that these are moduli of $p$-divisble groups, or formal group laws, which parametrize complex multiplicative cohomology theories.

3.1. **From motivic homotopy of schemes to automorphic forms.** A central problem in the Langlands Program is the conjectural relation between motives and automorphic forms [4]. This is exemplified by the Modularity Theorem, where to every elliptic curve over $\mathbf{Q}$ one can associate a modular form on $GL_2$. More generally, one expects that to every motive of a nonsingular, projective variety over $\mathbf{Q}$ is associated an automorphic form on $GL_n$ for some $n$.

While the general yoga of motives, first envisioned by A. Grothendieck, is still not fully developed, one particularly successful approach has been the motivic homotopy theory of schemes, developed by A. Morel and V. Voevodksy. The resulting stable homotopy category has in fact a model structure, which is a presentation of an underlying $(\infty, 1)$-category. The $(\infty, 1)$-category of motives is an active area of study, and more recently in the noncommutative setting [6]. Given Langlands' conjectures, the category of automorphic forms should reflect a similar structure.

Find a natural $(\infty, 1)$-category containing the category of automorphic forms on $GL_n$.

We already have several clues: M. Kapranov [3] and A. Parshin [5] have given a framework for an $n$-dimensional Langlands correspondence, which associates $n$-representations (in the sense of $n$-categories) to motives over $n$-dimensional local fields. Kapranov applies Waldhausen's $S$-construction to endow the category of motives with the structure of a bi-simplicial category, and formulates the Langlands correspondence as combinatorial stack on this category. In particular, the resulting bi-simplicial category is very close to that of an $(\infty, 1)$-category, hence the Langlands correspondence as formulated in Kapranov should carry over quite easily.

But Kapranov's construction is not ideal for the reason that the notion of an $n$-category is not well-understood, for say $n > 3$. Thus the category of $n$-vector spaces in which Kapranov's stack takes values in is difficult to work with in general. Instead, following work of Lurie it is easier to work with the limiting case of $(\infty, 1)$-categories. Here J. Bernstein's note [2], which offers the category of sheaves on algebraic stacks $BG$ rather than the category of representations of reductive groups $G$ as the right perspective for the Langlands correspondence in the sense of taking into account pure inner forms. Our first place to look, then, is to replace this category of sheaves with simplicial presheaves, following work of R. Jardine and others, which may lead us to the right model structure for an $(\infty, 1)$-category of automorphic forms.

3.2. **The spectrum of topological automorphic forms.** In this section, we introduce the spectrum defined by Behrens and Lawson using the spectrum of topological automorphic forms. Throughout we will work over a fixed a prime $p$, and a finite set of places $S$ of $\mathbf{Q}$.

3.3. **Outline of Behrens' proof.** Define the spectrum $Q(l)$, and use its building resolution to define a finite cochain complex $C^\cdot(l)$ of modular forms whose cohomology gives $\pi_* Q(l)$, with the differential given in terms of the Verschiebung operator. Analyze the chromatic spectral sequence of $Q(l)$, showing that the $E_1$ term consists of three lines $M_0 Q(l)$, $M_1 Q(l)$, and $M_2 Q(l)$, and relate $\pi_* M_2 Q(l)$ to the beta family.

Keep in mind the following communications to the author (reproduced without permission):

- Lawson: The action of the Hecke operators has been hard to determine; more seriously the dimension of the building complex at a prime ell tends to be not as high as we'd like.
- Behrens: I believe, at least on the level of the chromatic spectral sequence 0-line, this should happen ($Q_U(1, n-1)$ should detect greek letters) and this should follow from the density result using strong approximation that we

prove in our TAF memoir. The real work is coming up with an arithmetic interpretation of what this means. Also (and this seems tricky) to show the resulting chromatic spectral sequence classes persist to the $E_2$-term of the ANSS for the $Q$-spectra.

3.4. **Topological automorphic forms.** Define the spectrum of topological automorphic forms as the global section

$$TAF(K^p) = \mathcal{E}(K^p)(Sh(K^p)^\wedge_p)$$

where $\mathcal{E}(K^p)$ is a presheaf of $E_\infty$-ring spectra on the étale site of the completion at $p$ of the Shimura stack $Sh(K^p)$ satisfying properties [1, p.56]. Associated to the spectrum is the descent spectral sequence

$$H^s(Sh(K^p)^\wedge_p, \omega^{\otimes t}) \Rightarrow \pi_{2t-s}(TAF(K^p))$$

where $\omega$ is a line bundle defined by invariant 1-forms on the formal part of the 1-dimensional summand of the $p$-divisible group $A(p)$.

We may also recover this spectrum by taking $K^p$ homotopy fixed points of the smooth $GU(\mathbf{A}^{p,\infty})$-spectrum

$$V_{GU} := \operatorname*{colim}_{K^p} TAF(K^p)$$

so that for any compact open subgroup $K^p$ of $GU(\mathbf{A}^{p,\infty})$, there is an equivalence $TAF(K^p) \simeq V_{GU}^{hK^p}$.

Define the subgroup

$$K^{p,S}_{1,+} := GU^1(\mathbf{A}_S)K^{p,S}$$

where $GU^1(\mathbf{A}_S)$ consists of $g$ whose similitude norm has valuation 0 at every $v$ in $S$. We then define the spectrum $Q_U(K^{p,S})$ to be the homotopy fixed point spectrum

$$Q_U(K^{p,S}) := V_{GU}^{hK^{p,S}_{1,+}}$$

analagous to the above. There is an equivalence of $K(n)$ local spectra [1, 14.5.6]

$$TAF(K^p)_{K(n)} \simeq \Big(\prod_g E_n^{h\Gamma(gK^pg^{-1})}\Big)^{\mathrm{hGal}}$$

where the product runs over equivalence classes $g$ in $\Gamma\backslash GU^1(\mathbf{A}^{p,\infty})/K^p$.

3.5. **The semi-cosimplicial resolution.** Consider the Bruhat-Tits building $\mathcal{B} = \mathcal{B}(U(\mathbf{Q}_l))$ on which $GU^1(\mathbf{Q}_l)$ acts simplicially with open compact stabilizers [1, Lemma 12.3.5].

By [1, Theorem 13.2.9] there exists a semi-cosimplicial resolution spectrum $Q^\cdot$ of length $\dim \mathcal{B}$ whose $i$-th term is given by

$$Q^i = \prod_{[\sigma]} TAF(K^{p,l}K_{l,\sigma})$$

where the product runs over $GU^1(\mathbf{Q}_l)$-orbits of $i$-simplices of $\mathcal{B}$ and $K_{l,\sigma}$ is the stabilizer of $\sigma$.

## 4. Topological preliminaries

4.1. **Simplicial sets.** Let $\Delta$ be the category whose objects are the relations $0 \to 1 \to \cdots \to n$ for $n \geq 0$, and the morphisms are simply order-preserving set functions. Then a *simplicial set* is a contravariant functor

$$X : \Delta^{\mathrm{op}} \to \mathbf{Sets}.$$

The basic examples are the standard $n$-simplices $\Delta^n$ in $\mathbf{R}^{n+1}$, in the category of simplicial sets **sSets** we define it as

$$\Delta^n = \mathrm{Hom}_\Delta(\cdot, n)$$

the contravariant functor on $\Delta$ represented by $n$.

Given a (small) category $C$, the *nerve*, or *classifying space* of $C$, is the simplicial set with

$$BC_n = \mathrm{Hom}(n, C),$$

the set of functors from $n$ to $C$, i.e., strings of composable arrows of length $n$ in $C$. Technically, what we really mean by the classifying space is the realization $|BC|$, the topological space functorially associated to $BC$, but this will not matter from the point of view of homotopy theory.

A discrete group $G$ can be viewed as a groupoid—hence a category—with one object $*$ and one morphism $g : * \to *$ for every $g$ in $G$. Then $|BG|$ is the Eilenberg-MacLane space $K(G, 1)$.

A *simplicial presheaf* is a contravariant functor $X : C^{\mathrm{op}} \to \mathbf{sSets}$. etc. etc.

4.2. **The closed model structure.** A *closed model category* is a category $C$ with three classes of maps: fibrations, cofibrations, and weak equivalences, such that the following axioms are satisfied:

1. $C$ is closed under finite limits and colimits.
2. If $g$ is a fibration, cofibration, or weak equivalence, and $f$ is a retract of $g$, then so is $f$.
3. Given a commutative diagram [TRIANGLE HERE], if any two $f, g$, or $h$ are weak equivalences, then so is the third.
4. Given a commutative diagram [SQUARE HERE] where $i$ is a fibration and $p$ a cofibration, then the diagonal arrow exists, making the diagram commute, if either $i$ or $p$ is also a weak equivalence.
5. Any map $f : X \to Y$ may be factored as
   (a) $f = p \cdot i$ where $p$ is a fibration and $i$ is a trivial cofibration,
   (b) $f = q \cdot j$ where $q$ is a trivial fibration and $j$ is a cofibration.

**Theorem 4.2.1** ([**?**]). *The category of simplicial sets* **sSets** *with the Kan fibrations, cofibrations, and weak equivalences forms a closed model category.*

## 5. Towards a homotopy theory of automorphic forms

## References

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