ALGEBRAIC GENUS FOR SCHEMES

1. Motivation

Consider Ω_*^U, the complex cobordism ring of smooth oriented compact manifolds, isomorphic to the Lazard ring L. Define the genus (of a multiplicative sequence) as a ring homomorphism

$$\varphi : \Omega_*^U \otimes \mathbf{Q} \to R$$

where R is a $\mathbf{Q}$-algebra. Associated to such a φ is a power series $Q(x)$, called the characteristic power series of φ. By Thom there is a bijection between genera and formal power series with $\mathbf{Q}$ coefficients and leading coefficient 1.

Various genera have been defined, the most important being:

(1) The Todd genus, whose values are defined by Chern classes. In this case $Q(x) = z/(1 - e^{-z})$.
(2) The L-genus, , whose values are defined by Pontrjagin classes. By Thom, the signature of a manifold is given by a linear combination of Pontrjagin numbers, and moreover the signature theorem of Hirzeburch obtains the signature of M using the L-genus and the fundamental class of M.
(3) The $\hat{A}$-genus, whose values are also defined by Pontrjagin classes. It is related to the index of the Dirac operator, and in particular the Atiyah-Singer index theorem.
(4) The elliptic genus, generalizes (2) and (3) above. Ochanine showed that a genus is ellpitic iff its logarithm satisfies
$$\int_0^u \frac{dt}{\sqrt{1 - 2\delta t^2 + \epsilon t^4}}$$
for constants δ, ϵ in R. This is an elliptic integral, hence the name. It gives rise to the elliptic cohomologies.
(5) The Witten genus, whose characteristic power series is defined using Eisenstein series, was shown to be the universal elliptic genus. Taking values a priori in $\mathbf{Q}[[u]]$, it was shown that it in fact takes values in $M_*(\Gamma_0(2))$, the ring of modular forms for $\Gamma_0(2)$.

This genus can be upgraded to a homomorphism of E_∞-ring spectra. By the Witten genus we are led to

$$MU \to TMF$$

and

$$M\text{String} \to tmf.$$

Now Levine and Pandharipande have constructed a geometric algebraic cobordism theory ω_*, such that over a field of characteristic 0,

$$\omega_*(X) \simeq MGL^{2*,*}(X)$$

where X is any smooth scheme over k, and MGL is the algebraic cobordism constucted by Voevodsky. In particular, $MGL^{2*,*}(\mathrm{Spec}(k)) \simeq L$. We'd like to extend the notion of a genus to schemes, namely, to ω_*, and possibly to MGL in general.

2. Double point cobordism

To define the ring ω_*, we need to first define the double point relation. Throughout all schemes will be considered to be smooth and quasiprojective over a field k of characteristic 0, and denote this category by $\mathbf{Sm}_k$. The following definition is adapted from Levine and Pandharipande.

Definition 1. *(Double point relation) Let $X \in \mathbf{Sm}_k$. Let also $Y \in \mathbf{Sm}_k$ of pure dimension with projective morphisms $Y \to X$ and $\pi : Y \to \mathbf{P}^1$. We call*

$$[Y_\infty] - [Y_1] - [Y_2] + [Y_3]$$

a double point relation, if

1. *$Y_\infty = \pi^{-1}(\infty)$ is the fiber over any regular value ∞ in $\mathbf{P}^1$.*
2. *Y_1 and Y_2 are smooth Cartier divisors intersecting transversally on Y, such that $\pi^{-1}(0) = Y_1 \cup Y_2$.*
3. *$Y_3 = \mathbf{P}(O_D \oplus N_{X_1/D}) \simeq \mathbf{P}(O_D \oplus N_{X_2/D})$ the projective bundle over the intersection $D = X_1 \cap X_2$ and $N_{X_i/D}$ the normal bundle of D in X_i.*

Definition 2. *(Double point cobordism) Let $X \in \mathbf{Sm}_k$. Define $\omega_*(X)$ to be the free abelian group generated by projective morphisms*

$$f : Y \to X$$

where Y is any irreducible smooth k-scheme up to equivalence.

In fact, $\omega_*(X)$ is a ring under disjoint union and cartesian product, graded by dimension.

Now let $\Omega_*(X)$ be the theory of algebraic cobordism defined by Levine and Morel. By Levine and Pandharipande there is a canonical isomorphism

$$\omega_*(X) \simeq \Omega_*(X).$$

In particular, $\omega_*(\mathrm{Spec}(k)) \simeq L$ where L is the Lazard ring, and

$$\omega_*(\mathrm{Spec}(k)) \otimes \mathbf{Q} = \mathbf{Q}[\mathbf{P}^1, \mathbf{P}^2, \dots].$$

Now we can introduce the notion of an algebraic genus:

Definition 3. *An algebraic genus is a ring homomorphism*

$$\varphi : \omega_* \otimes \mathbf{Q} \to R$$

where R is a $\mathbf{Q}$-algebra.

Alternatively, we may view φ as a function on schemes satisfying

1. $\varphi(Y \cup Z) = \varphi(Y) + \varphi(Z)$
2. $\varphi(Y \times Z) = \varphi(Y) \cdot \varphi(Z)$
3. $\varphi(Y_\infty) = 0$ if Y satisfies the double point relation.

So we'd like to check first that this algebraic genus so defined recovers the classical genera, for the reason that algebraic cobordism for schemes agrees with complex cobordism for complex manifolds.

2.1. **The Todd genus.** Since one has a theory of Chern classes for algebraic vector bundles, we already have the Todd genus given (really in Arakelov geometry): Let E be a rank n vector bundle with Chern roots $\alpha_1, \ldots, \alpha_n$, then the Chern character is defined as

$$\mathrm{ch}(E) = \sum_{i=1}^{n} \exp(\alpha_i)$$

while the Todd class is defined as

$$\mathrm{td}(E) = \prod_{i=1}^{n} Q(\alpha_i)$$

where $Q(z) = z/(1 - e^{-z})$ is the characteristic power series. This is exactly the same as in the complex case.

This genus is characterized by the property that it takes the value 1 on all the generators $\mathbf{P}^n$. In the complex case, this was shown using the Hirzebruch-Riemann-Roch formula. In our case, we appeal to the Grothendieck-Riemann-Roch:

$$\mathrm{ch}(f_! \mathcal{F}^{\cdot})\mathrm{td}(X) = f_*(\mathrm{ch}(\mathcal{F}^{\cdot})\mathrm{td}(Y)$$

where $f : Y \to X$ is a proper morphism of smooth quasi-projective schemes, $\mathcal{F}^{\cdot}$ is a bounded complex of sheaves on Y. Here

$$f_! = \sum_i (-1)^i R^i f_*,$$

and f_* is the pushforward on the Chow ring. We use this to evaluate the Todd class on $\mathbf{P}^n$ to get 1 for any n.

Proposition 1. *The Todd genus is invariant under algebraic cobordism.*

To prove this one only has to show that Chern classes are algebraic cobordism invariant.

2.2. **The L-genus.** To define the L-genus and $\hat{A}$-genus we need a theory of algebraic Pontrjagin classes. How to do this? The first thing we might try is the following: the k-th Pontrjagin class of a real vector bundle E is defined by the Chern class of the complexification

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbf{C}) \in H^{4k}(M, \mathbf{Z}).$$

So we might define algebraic Pontrjagin classes again in terms of algebraic Chern classes directly as in the second expression above.

Definition 4. *Let E be an algebraic vector bundle over X. Define the algebraic Pontrjagin class $p_k(E) = (-1)^k c_{2k}(E)$. Alternatively, we may follow Panin and define algebraic Pontrjagin classes of symplectic bundles. They are $\mathbf{A}^1$-deformation invariant and nilpotent.* [1]

Moreover, for the L-genus we also need to generalize the signature of a manifold to schemes: given a connected oriented manifold M of dimension $4k$, the cup product on middle cohomology

$$\smile : H^{2k}(M, \mathbf{Z}) \times H^{2k}(M, \mathbf{Z}) \to H^{4k}(M, \mathbf{Z}) \simeq H_0(M, \mathbf{Z}) \simeq \mathbf{Z}$$

[1] See Panin and Walter, Quaternionic Grassmanians and Pontryagin classes in algebraic geometry

in the case where Poincaré duality holds, gives rise to a quadratic form on $H^{2k}(M, \mathbf{Z})$. Then the signature of M is defined to be the signature of the quadratic form. If M has dimension prime to 4, then the signature is defined to be 0. Then the Hirzebruch signature theorem gives the signature of M as the L-genus of M evaluated on the fundamental class of M.

In the case of schemes, we should like to use either étale cohomology or motivic cohomology. The Poincaré duality pairing on middle cohomology in each case will again lead us to a quadratic form.

2.3. **The $\hat{A}$-genus.** This genus was shown to be a modular form, involving the classical Dedekind eta function. On the other hand, it is related to the index of the Dirac operator, and more generally the Atiyah-Singer index theorem. Note that the equivariant index theorem is in fact related to the Lefschetz fixed point theorem.

Again, this genus being defined by Pontrjagin classes, we are back in the discussion of the previous section.

3. Algebraic cobordism

Generalizing the notion of genera to the double-point cobordism seems fairly routine, since there one would like to recover basically the same results. In particular, we'd get a similar map to modular forms, nothing more. The interesting part is to speculate how one might extend this to the entire MGL, in which case one obtains more than the Lazard ring, and hence the image of this putative extended algebraic genus

$$\varphi : MGL \otimes \mathbf{Q} \to R$$

will land in something bigger than modular forms (first of all, bigraded), perhaps leading to the notion of MMF.

Remark 1. *Why might this be interesting to a number theorist? From the perspective of the Langlands Program, one would like to bring closer the world of motives and modular forms. In particular, the conjectural Tannakian formalism on the category of (Grothendieck) motives and (isobaric) automorphic forms furnishes a map of motivic Galois groups*

$$G_F \to L_F$$

where the group on the right is the conjectural automorphic Langlands group, an much desired object in the classification of automorphic representations of reductive algebraic groups. While this formalism is not available, one might ask if we might 'represent' the above categories in their stable versions, namely, by means of ring spectra.